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THE TERCENTENARY EXHIBITION OF NAPIER RELICS

AND OF BOOKS, INSTRUMENTS, AND DEVICES FOR FACILITATING CALCULATION

MODERN INSTRUMENTS
AND METHODS
OF CALCULATION

A HANDBOOK OF THE

NAPIER TERCENTENARY EXHIBITION

EDITED BY

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Preface

The aim of the Exhibition is to do honour to one whose influence on science has been singularly profound; partly by a display of relics, partly by indicating the scope of his work, but more particularly by tracing what may be considered as the development of his great achievement. The modern mathematical laboratory may look upon Napier as its parent.

An endeavour has been made to make the Exhibition and Handbook useful to the laboratory computer, the engineer, the astronomer, the statistician, and to all who are interested in calculation.

The Editor desires to express his grateful thanks to many helpers: to Professor Whittaker and Dr Knott for help both with the general scheme and also in the details; to the writers of the articles; and to the lenders of the exhibits.

He also takes this opportunity of acknowledging the valuable services of his colleague Mr Gibb, who has assisted him in the revision of the proof sheets.

A special acknowledgment is fitting to Principal Sir William Turner, K.C.B., F.R.S., and to the Members of the University Court, who granted the use of rooms in the University for the Exhibition; and to Sir T. Carlaw Martin, LL.D., Director of the Royal Scottish Museum, who kindly lent the cases in which the exhibits were displayed.

The closing days of the preparation were overshadowed by the death of a valued contributor, Mr John Urquhart, M.A., Lecturer in Mathematics in the University of Edinburgh. One of the three articles which Mr Urquhart wrote for the present work, in collaboration with Dr Carse, was still unfinished when he was attacked by the malady which was to prove fatal. To his many friends these writings will be a memorial of one whom they will ever remember with admiration and affection.

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The Handbook of the Napier Tercentenary Exhibition

Section A

NAPIER'S LIFE AND WORKS

Napier and the Invention of Logarithms. By Professor George A. Gibson, M.A., LL.D.

(Reprinted from the *Proceedings* by permission of the Royal Philosophical Society of Glasgow.)

In 1614 John Napier of Merchiston published his *Description of the Admirable Canon of Logarithms*.[1] On the title-page of the book he is called their "author and inventor," and the words were the simple statement of a fact, because he was the inventor both of the method of logarithmic calculation and of the word logarithm itself.

At the present day it is perhaps somewhat difficult to form an adequate conception of the greatness of Napier's invention; yet it is beyond all question that the invention of logarithms marks an epoch in the history of science. It is generally admitted that Newton's *Principia* is one of the great works that have shaped the course not merely of modern science in its practical aspects, but of scientific thought in relation to philosophy and theology. But the debt of Newton to Napier, though indirect, was very real, because Newton was essentially dependent on the results of Kepler's calculations, and these calculations might not have been completed in Kepler's lifetime but for the aid that the logarithms afforded. Kepler felt keenly the grievous burden imposed upon him by the older methods, and was correspondingly gratified by the relief that the new means of calculation provided. Without the logarithms or some similar help astronomical observations could only have been reduced, if at all, with the very greatest difficulty, and the development of modern science might have followed a very different course.

The significance of Napier's invention becomes all the more remarkable when we consider the condition of Scotland during his lifetime. Throughout the greater part of the sixteenth century there was incessant unrest, and such intellectual interests as made themselves felt were predominantly associated with ecclesiastical and theological discussions. Though the foundation of the University of Edinburgh in 1582 increased the number

[1] For the original Latin title, see p. 8.

of Universities to four, the higher learning could hardly be said to "flourish," even if we allow for the energetic principalship of Andrew Melville. The instruction given in the Universities was necessarily of a very elementary kind, since adequately prepared students were not being sent up from the schools, and there was no scientifically minded public to whom appeal might be made. Before Napier, Scotland made not a single contribution to mathematical science, and the appearance early in the seventeenth century, in *Scotland*, of a book that at once took rank as one of the great landmarks of scientific discovery has been a constant subject of remark by the historians of mathematics.

It may be true that, in the language of Professor Hume Brown (*History*, ii. 280), "at the beginning of the sixteenth century Scotland could more than hold its own with England in the number and quality of its men of literary genius"; yet it is literary and not scientific eminence that is here claimed. It is, however, the second half of the sixteenth and the opening of the seventeenth century that is of more immediate importance in estimating Napier's environment, and the same historian states that during this period the relation between the two countries was signally reversed. However eminent Buchanan may have been, there is no one, I suppose, who claims for him a place as an exponent of mathematical or physical science, so that Napier's appearance as a mathematician of the highest rank is probably unique in scientific history.

John Napier, the inventor of logarithms, was born in 1550, at Merchiston Castle, near Edinburgh. Though he must have spent a considerable part of his life on the Lennox and Menteith estates of his family, and had a residence at Gartness, the tradition that claims Gartness for his birthplace must be abandoned. Such knowledge as we possess of Napier's private life is due almost entirely to the industry of his descendant, Mark Napier, whose *Memoirs of John Napier of Merchiston: His Lineage, Life, and Times* (Edinburgh, 1834) is based on careful research, especially of the private papers of the Napier family, and is the source of all modern accounts. As a biography the *Memoirs* cannot be assigned a high place in that branch of literature; the narrative is encumbered with wearisome digressions on the Napier connections, and is not free from that prejudiced view of the history of the period which is still so common. The hero-worship of the biographer seems to extend to the whole Napier family, and becomes monotonous, if not repellent; occasionally, as when the Presbyterian leanings of the Napiers come into conflict with the policy of their sovereigns, the biographer has a hard struggle to reconcile the divergent loyalties. It would, however, be ungrateful to insist too much on the defects of a biography which has brought together so much that is really valuable.

John Napier was the eighth Napier of Merchiston. According to the *Memoirs*, Alexander Napare, the first of Merchiston, acquired that estate before the year 1438 from James I., was Provost of Edinburgh in 1437, and was otherwise distinguished in that reign. His eldest son, also Alexander, became in his father's lifetime Comptroller to James II., and "ran a splendid career under successive monarchs." The origin of these ancestors of John Napier is very uncertain. In the thirteenth and fourteenth centuries persons

of the name of Napier were not uncommon, especially in the Lennox. The Merchiston family cherished a tradition that their name was changed from Lennox to Napier by command of a king of the Scots who wished to do honour to one of their ancestors, Donald, a son of an Earl of Lennox. This Donald, it is said, had turned the tide of battle when flowing strongly against the king, and had fought so valiantly that the king declared before all the troops that he had Na Peer. The name is probably of a more domestic origin, and commemorates virtues that are not usually associated with the warrior, though the "punning" or "canting" derivation of the name is fairly frequent in connection with the great Napier. On one occasion he is quoted, quite seriously it would seem, as "un Gentilhomme Ecossois nommé Nonpareil"; and one of the commendatory odes prefixed to the *Canon Mirificus* of 1614 ends with these lines:—

> "Nomine sic Nepar Parili fit et omine Non Par,
> Quum non hac habeat Nepar in arte Parem."

It is perhaps of more importance that we do not know the correct spelling of Napier's name, since many forms of the word are found, such as Napeir, Nepair, Nepeir, Neper, Napare, Naper, Naipper. Apparently the forms Jhone Neper and Jhone Nepair are the most usual with John Napier; the form Napier is said to be comparatively modern.

The Merchiston family had close associations with Edinburgh, and several of its members were provosts of the city. During the fifteenth and sixteenth centuries the Napiers of Merchiston formed numerous alliances with noble families, and acquired extensive estates in the Lennox and Menteith; they held various offices connected with the royal household, and, so far as I can make out, were able to keep what they had won whatever faction was in power.

Sir Archibald Napier, the father of John, was the son of an Alexander Napier who fell at Pinkie, and at the time of his father's death had not completed his fifteenth year. On the 8th of November 1548, he obtained a royal dispensation enabling him, though a minor, to feudalise his right to his paternal barony, and in the following year, when he was only fifteen, he married Janet Bothwell, the daughter of an Edinburgh burgess. Archibald Napier had the usual fortune of the family. He received the honour of knighthood in 1565, and about 1582 was appointed Master of the Mint with the sole charge of superintending the mines and minerals within the kingdom—an office he held till his death in 1608. In 1561 he appears as a justice-depute. In a register from 17th May 1563 to 17th May 1564, the justice-deputes named are "Archibald Naper of Merchiston, Alexander Bannatyne, burgess of Edinburgh, James Stirling of Keir, and Mr Thomas Craig." John Napier was married to Stirling's daughter, and was an intimate friend of Dr John Craig, the son of Thomas Craig.

Janet Bothwell was the sister of Adam Bothwell, the Bishop of Orkney; her mother, Katherine Bellenden, was thrice married, her third husband being Oliver Sinclair, the favourite of James V., and it was in Sinclair's house that she was brought up.

When John Napier was born his parents must have been very young, not more than sixteen. Of his boyhood and early education very little is known;

the only reference on record occurs in a letter of date 5th December 1560, when he was about ten years old, to his father from the Bishop of Orkney, Adam Bothwell. The letter contains the following passage :—

" I pray you, schir, to send your son Jhone to the schuyllis ; oyer to France or Flandaris ; for he can leyr na guid at hame, nor get na proffeitt in this maist perullous worlde—that he may be saved in it,—that he may do frendis efter honnour and proffeitt as I dout not bot he will : quhem with you, and the remanent of our successioune, and my sister, your pairte, Got mot preserve eternalle."

It is possible that the bishop had already detected indications of the genius that was later to become so manifest, but it seems to me more likely that the interest shown in the son was intended to stimulate the father to exert himself on the bishop's behalf in certain legal proceedings which form the main subject of the letter.

In 1563 John Napier's mother died, but before her death he had matriculated at St Salvator's College, St Andrews, and, by an arrangement made apparently by his mother, he was boarded within the college under the special charge of the principal, John Rutherfurd. Of the students whose names occur on the matriculation roll of St Salvator's for 1563, there is none except Napier himself who was afterwards distinguished as scholar, preacher, or statesman. Had Napier followed the usual course his name would appear in the list of *Determinantes* for 1566, and of Masters of Arts for 1568 ; but no trace of it has been found, and the only conclusion to be drawn from its absence is that his residence at St Salvator's was comparatively short. Principal Rutherfurd seems to have been a man of respectable attainments, but there can be little doubt that it was not at St Andrews that Napier acquired his wide knowledge of classical literature or was set upon the path that led to his discoveries and inventions in the field of mathematics.

The influence on his future life of his residence at St Andrews was, nevertheless, of the most far-reaching character ; for it was then that he received an impetus to theological studies that formed throughout his life quite as great an attraction as mathematics in any of its branches. He himself tells the story in the address " To the Godly and Christian Reader " prefixed to his first publication, *A Plaine Discovery of the Whole Revelation of St John*. In that address we find the following passage :—

"Although I have but of late attempted to write this so high a work, for preventing the apparant danger of Papistry arising within this Island ; yet in truth it is no few yeers since first I began to precogitate the same : For in my tender yeers and barneage at Saint Androes at the Schools, having on the one part contracted a loving familiarity with a certain Gentleman, &c., a Papist ; and on the other part being attentive to the Sermons of that worthy man of God, Master Christopher Goodman, teaching upon the Apocalypse, I was so moved in admiration against the blindnesse of Papists that could not most evidently see their seven-hilled-city, Rome, painted out there so lively by Saint John, as the Maker of all Spiritual Whoredom, that not only burst I out in continuall reasoning against my said familiar, but also from henceforth I determined with myself (by the assistance of God's spirit) to employ my studie and diligence to search out the remanent

Photo by A. Swan Watson.

MERCHISTON CASTLE FROM THE EAST.

By permission of George Smith, M.A., Headmaster of Merchiston Castle.

[*To face p.* 68.

mysteries of that holy Book; as to this hour (praised be the Lord) I have been doing, at all such times as I might have occasion."

Theology of course bulked largely in the discussions of the sixteenth century, and it seems to have had a fascination for Napier. Various references in his mathematical works can only be explained on the assumption that he could not divert his attention from theological studies sufficiently long to enable him to carry out cherished mathematical investigations. Whatever we may think of the ascendency that James VI. acquired over the Church in Scotland, I am inclined to believe it is James's victory over the Presbyterian party, to which Napier belonged, that compelled Napier to withdraw from the ecclesiastical field and devote himself to his mathematical studies.

It is almost certain, though there is no explicit documentary evidence, that Napier after leaving the University followed the advice of Adam Bothwell and spent some years on the Continent, studying probably at the University of Paris and visiting the Netherlands and Italy. The extreme probability that, as a member of a noble family, he would be sent to pursue his studies abroad is confirmed by some interesting facts which are mentioned by Mark Napier in his introduction to the posthumous work *De Arte Logistica*. Of Napier's travels or of the men under whom he studied or with whom he made acquaintance we have, however, no record; we know that he was in Scotland in 1571. In that year his father married again, his second wife being Elizabeth Mowbray, a daughter of John Mowbray of Barnbougall, and the sons of this marriage were at a latter date the cause of considerable anxiety and worry to their half-brother.

Towards the end of 1571 negotiations were begun for Napier's marriage to Elizabeth Stirling, daughter of Sir Archibald's old friend and fellow justice-depute, Sir James Stirling of Keir. The marriage did not take place for some time as Sir Archibald had become involved in political troubles; indeed, the exact date of the marriage does not seem to be known, though it probably took place in the end of 1572 or early in 1573. A royal charter of date 8th October 1572, granted to Napier and his future wife in conjunct fee the lands of Edinbellie, the two Ballats, Gartness, etc., in the barony of Edinbellie-Napier; also to John Napier the lands of Merchiston and the Pultrielands. The liferent of all, except the lands in conjunct fee, was reserved to Sir Archibald and his wife, Elizabeth Mowbray. A castle, completed in 1574, was built at Gartness, and there John Napier and his wife took up their residence.

Of the details of Napier's life at Gartness we know nothing beyond vague traditions. He did not succeed his father till 1608, and, though he was now Fear of Merchiston, Gartness must have been his home. The management of the Napier estates in the Lennox and Menteith evidently occupied much of his time, but he seems to have been frequently in Edinburgh, and of the few letters printed in the *Memoirs* there is none dated from Gartness, though there is one from Keir. The dedication of his commentary on the Revelation is dated "at Marchistoun, the 29 day of January 1593." It is somewhat singular that Gartness figures so little in any record we have of him.

Napier's first wife died in 1579, leaving one son, Archibald, the first Lord Napier, and one daughter, Jane. From among his own relations, but, in the language of his biographer, from "a family deeply dyed in scarlet," he took a second spouse, Agnes Chisholm, daughter of Sir James Chisholm of Cromlix; by her he had ten children, five sons and five daughters. It is in connection with proceedings in which Sir James Chisholm figured prominently that Napier first appears in the public life of Scotland.

Napier, as we have seen, was deeply interested, even during his St Andrews days, in the religious questions that formed the subject of such keen controversy; he allied himself with the Protestant party, and maintained a close friendship with the Edinburgh ministers. When the Church took action in the affair of the Spanish Blanks, he was one of the commissioners appointed at a meeting held at Glasgow, on 11th October 1593, to meet at Edinburgh with commissioners from the other districts of Scotland to give advice and counsel as to procedure. Napier attended the convention at Edinburgh on the 17th of October, and joined in the excommunication then pronounced of his father-in-law, Sir James Chisholm. The convention appointed a committee, of whom Napier was one, to seek an interview with the king, and press on him certain measures for the safety of the Church and the punishment of the rebels. Napier and his colleagues after some difficulty secured the desired interview, but the net result of their labours can hardly have been satisfactory to them. James proved to be too strong for the Church, and there is no record of any further protest on Napier's part.

The fears entertained at this time in Scotland of an invasion by Philip of Spain had aroused Napier's anxiety for the cause of Protestantism, and he published in January 1593–4 the book already referred to—*A Plaine Discovery of the Whole Revelation of St John*. A second edition, revised and enlarged, was published in 1611, and the book continued to be republished for several years. It was also translated into Dutch, French, and German. The French translation was executed by a Scotchman named George Thomson, and is said on the title-page to have been revised by Napier himself. The dedication to King James contains some plain speaking about the duties of kings, princes, and governors in their relations to the Church; and the whole treatment of the subject is based on presuppositions that are accepted by very few at the present day. There is good evidence for the belief that this commentary secured for Napier, not merely at home but even more markedly on the Continent, the reputation of a scholar and theologian of high rank. But I suppose that there are few indeed of the present generation who have read, or have even heard of, the book; whatever its merits may have been they do not appeal to the modern mind, and in any case I do not feel competent to set them forth.

It may not be out of place to remark that at the end of the treatise are added "certain oracles of Sibylla"; Napier quotes them from Castalio's Latin translation, but presents them to his readers in English verse. There is a terseness and a rhythm in the lines that are not usually found in translations, and that bear out the supposition that Napier was not merely an accurate scholar but had a touch of poetic genius.

Perhaps Napier's authority as a divine saved him from persecution as a

warlock. Traditions that he was in league with the powers of darkness might, it is said, be met with in the cottages and nurseries in and about Edinburgh not very many years ago. Among these traditions is one of a jet-black cock which was his constant companion, and was supposed to be a familiar spirit bound to him in that shape. Mark Napier takes the story of the cock so seriously that he tries to *rationalise* the tradition by suggesting that Napier played upon the belief in his witchcraft to frighten his servants into confession of misdemeanours. But the soot-bedaubed cock and the intoxicated pigeons hardly deserve serious mention.

From the parish of Killearn come other traditions. In the *Statistical Account*, vol. xvi. p. 108, we find the following reference to Napier :—" Adjoining the mill of Gartness are the remains of an old house in which John Napier of Merchiston, Inventor of Logarithms, resided a great part of his time (some years) when he was making his calculations. It is reported that the noise of the cascade, being constant, never gave him uneasiness, but that the clack of the mill, which was only occasional, greatly disturbed his thoughts. He was therefore, when in deep study, sometimes under the necessity of desiring the miller to stop the mill that the train of his ideas might not be interrupted. He used frequently to walk out in his nightgown and cap. This, with some things which to the vulgar appeared rather odd, fixed on him the character of a *warlock*. It was formerly believed and currently reported that he was in compact with the devil; and that the time he spent in study was spent in learning the *black art* and holding conversation with *Old Nick*."

These traditions are in harmony with a superstitious age, but it seems to be beyond question that Napier was not free from a belief in some forms of magic. A curious document has been preserved which records, under date July 1594, an agreement with the notorious Logan of Restalrig to exert his powers in the search for some treasure supposed to be hidden in Logan's keep of Fast Castle. It is doubtful if the trial actually took place, but the agreement is written in Napier's own hand and is certainly genuine. Napier, however, soon broke with Logan; the only wonder is that he ever had friendly dealings with him. It is a testimony to the high respect in which Napier was held that he does not seem to have been challenged at any time as the possessor of magical powers; in that age, even his rank would not have protected him had he been charged with being in league with the prince of darkness.

As the possessor of extensive estates it is fitting that Napier should have turned his attention to the improvement of agriculture. He took keen interest in his property, was inclined to insist upon what he thought to be his rights, but was at the same time eager to promote methods of tillage that offered prospects of better returns. He is said to have carried out careful experiments on " the gooding and manuring of all sorts of field land with common salts, whereby the same may bring forth in more abundance, both of grass and corn of all sorts, and far cheaper than by the common way of dunging used heretofore in Scotland."

Napier's inventiveness was not limited to the peaceful domain of mathematics, but showed itself in devising instruments of war. Mark Napier gives a facsimile of a document preserved in the Bacon Collection in Lambeth

Palace, in which John Napier describes some " Secret Inventions, profitable and necessary in these days for defence of this Island and withstanding of strangers, enemies of God's truth and religion." The inventions consist of (1) a mirror for burning the enemies' ships at any distance, (2) a piece of artillery destroying everything round an arc of a circle, and (3) a round metal chariot so constructed that its occupants could move it rapidly and easily, while firing out through small holes in it. Sir Thomas Urquhart asserts that Napier did construct an engine which he tested on a large plain in Scotland " to the destruction of a great many herds of cattle and flocks of sheep, whereof some were distant from other half a mile on all sides, and some a whole mile." It would be hazardous, however, to make any assertion on the strength of Sir Thomas's evidence, and we know too little about these inventions to form any definite conception of them ; but there is little doubt that Napier had quite decided mechanical skill.

Of Napier's claims to remembrance, however, the greatest is his invention of logarithms. It has often been remarked that the great discoveries and inventions have always come just when the time was ripe for them, and that if one man had not made the decisive step in advance another would have done so almost as soon. This statement is perhaps less accurate in regard to the invention of logarithms than in respect of many other discoveries ; for, with one possible exception, there is no suggestion even that Napier has a rival. The exception is Jobst Bürgi, an ingenious Swiss watchmaker and mechanic. But Napier's *Canon Mirificus* was published six years before Bürgi's *Progress Tabulen* ; Bürgi's Tables are very imperfect compared with Napier's ; and there is every reason for believing that Napier had formed his conception of logarithms and begun their calculation quite as early as Bürgi—probably much earlier. Besides, Bürgi's work has not had the slightest influence, so far as can be traced, either on the theoretical or on the practical development of logarithms. Napier is therefore entitled to the full credit of an invention which ranks, in respect of its importance in the history of British science, as second only to Newton's *Principia*.

The full title of Napier's work, published in 1614, is :—*Mirifici Logarithmorum Canonis Descriptio, Ejusque usus in utraque Trigonometria ; ut etiam in omni Logistica Mathematica, Amplissimi, Facillimi, & expeditissimi explicatio. Authore ac Inventore, Ioanne Nepero, Barone Merchistonii, &c. Scoto.* Edinburgi, Ex officina Andreae Hart Bibliopolae. CIↃ. DC. XIV.

This is printed on an ornamental title-page. The work is a small-sized quarto, containing 57 pages of explanatory matter, and 90 pages of tables. A facsimile of the title-page is given in the *Memoirs* (p. 374).

An English translation of the *Descriptio* was made by Edward Wright and published in 1616, after the death of the translator, by his son, Samuel Wright. Napier, as stated in Samuel Wright's dedication to the " Right Honourable and Right Worshipful Company of Merchants of London trading to the East Indies," read the translation, and " after great pains taken therein gave approbation to it, both in substance and form." It is therefore perhaps not out of place to give Napier's own account of his invention as that is recorded in " The Author's Preface to the Admirable Table of Logarithms " ; it is a slightly modified version of the Preface in the original Latin edition, the

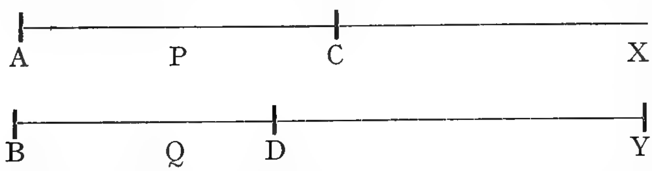
modification, however, referring merely to the general purpose and accuracy of the translation :—

"Seeing there is nothing (right well-beloved Students of the Mathematics) that is so troublesome to mathematical practice, nor that doth more molest and hinder calculators, than the multiplications, divisions, square and cubical extractions of great numbers, which besides the tedious expense of time are for the most part subject to many slippery errors, I began therefore to consider in my mind by what certain and ready art I might remove those hindrances. And having thought upon many things to this purpose, I found at length some excellent brief rules to be treated of (perhaps) hereafter. But amongst all, none more profitable than this which together with the hard and tedious multiplications, divisions and extractions of roots, doth also cast away from the work itself even the very numbers themselves that are to be multiplied, divided and resolved into roots, and putteth other numbers in their place which perform as much as they can do, only by addition and subtraction, division by two or division by three. Which secret invention, being (as all other good things are) so much the better as it shall be the more common, I thought good heretofore to set forth in Latin for the public use of mathematicians. But now some of our countrymen in this Island, well affected to these studies and the more public good, procured a most learned mathematician to translate the same into our vulgar English tongue, who, after he had finished it, sent the copy of it to me to be seen and considered on by myself. I having most willingly and gladly done the same, find it to be most exact and precisely conformable to my mind and the original. Therefore it may please you who are inclined to these studies to receive it from me and the translator with as much goodwill as we recommend it unto you. Fare ye well."

I do not think one can state more clearly the purpose of logarithms; a more detailed statement necessarily calls for treatment that belongs to the region of mathematics. To those who are only acquainted with logarithms as they are explained in the modern elementary text-books the following points may be of interest :—

1. Napier makes no use of a base. The conception of indices in the modern sense of fractional and negative indices was quite unknown in Napier's day and for long after. Algebra was as yet in far too crude a condition to provide a treatment of a logarithm as an index.

2. Napier's treatment is based on the comparison of the velocities of two moving points. Suppose one point P to set out from the point A and to move along the line AX with a uniform velocity V; then suppose another point Q to set out from B on



the line BY, *of given length r*, at the same time as P sets out from A and with the same velocity V as that of P on the line AX, but to move, not uniformly,

but so that its velocity at any point, as D, is proportional to the distance DY from D to the end Y of the line BY. If now C is the point that P has reached, moving with the uniform velocity V, when Q, moving in the way described, has reached D, then the number which measures AC is the logarithm of the number which measures DY.

Napier had the needs of trigonometry primarily in view, and he usually speaks of BY (or r) as the whole sine and DY as a sine; it will be remembered that in Napier's day the sine was a line and not a ratio as with us.

3. When Q is at B the other point P is at A, so that the logarithm of the whole sine BY is zero. The logarithms of numbers less than BY, say logDY, are positive numbers; if Q were to the left of B, then P would be to the left of A, and AP would be negative, so that in Napier's system the logarithms of numbers greater than the whole sine are negative.

The circumstance that log1 is not zero in Napier's system is very awkward. Napier was quite well aware of the disadvantages of taking the whole sine as the number whose logarithm was to be zero, and, as we shall see, afterwards suggested the change to a system in which log1 is zero. He had, however, some good reasons for his first choice, and it must be admitted that for the trigonometry he had chiefly in view the awkwardness is far less than it seems.

4. Napier next establishes the rule that if a is to b as c is to d, then

$$\log a - \log b = \log c - \log d,$$

and from this rule he readily establishes all the rules required for ordinary calculations.

The *Descriptio* besides stating and explaining the rules gives many examples of the use of logarithms in trigonometrical calculations of a most varied kind; in the course of the work he proves some valuable theorems in spherical trigonometry. The Tables give the sines, and the logarithms of the sines and of the tangents of all angles from 0° to 90° at intervals of one minute.

It is pleasant that we can state that the value of Napier's invention was at once recognised. As has been mentioned, an English translation appeared in 1616; this translation contains besides the author's own Preface one by Henry Briggs, Geometry-reader (or, as we would say, Professor of Mathematics) at Gresham College, London. Some interesting statements are preserved of the enthusiasm with which Briggs welcomed Napier's invention. In a letter to Archbishop Usher, dated at Gresham House 10th March 1615, he writes:—"Napper, lord of Markinston, hath set my head and hands a work with his new and admirable logarithms. I hope to see him this summer, if it please God, for I never saw a book which pleased me better or made me more wonder." Again, Dr Thomas Smith in his life of Briggs, in the "Vitæ quorundam eruditissimorum et illustrium virorum," says of him when describing his enthusiasm over the *Canon Mirificus*:—"He cherished it as the apple of his eye; it was ever in his bosom or in his hand, or pressed to his heart, and, with greedy eyes and mind absorbed, he read it again and again. . . . It was the theme of his praise in familiar conversation with his friends, and he expounded it to his students in the lecture room."

These expressions of Briggs are of special value to us at the present day,

because Briggs was a mathematician of great eminence; his appreciation of Napier's work gives us some definite conception, both of the grievous nature of the burden that necessary calculations imposed *on the really competent computer*, and of the relief that the logarithms provided.

Briggs paid his anticipated visit to Napier in the summer of 1615, and there is an interesting story told to Ashmole by William Lilly, the astrologer, of his reception at Merchiston. "I will acquaint you," Lilly narrates in his *Life*, "with one memorable story related unto me by John Marr, an excellent mathematician and geometrician whom I conceive you remember. He was servant to King James I. and Charles I. When Merchiston first published his Logarithms Mr Briggs, then reader of the astronomy lectures at Gresham College in London, was so surprised with admiration of them that he could have no quietness in himself until he had seen that noble person whose only invention they were. He acquaints John Marr therewith who went into Scotland before Mr Briggs purposely to be there when these two so learned persons should meet. Mr Briggs appoints a certain day when to meet at Edinburgh; but, failing thereof, Merchiston was fearful he would not come. It happened one day as John Marr and the Lord Napier were speaking of Mr Briggs, 'Oh! John,' saith Merchiston, 'Mr Briggs will not come now'; at the very instant one knocks at the gate, John Marr hasted down and it proved to be Mr Briggs to his great contentment. He brings Mr Briggs into my Lord's chamber, where almost one quarter of an hour was spent, each beholding other with admiration, before one word was spoken. At last Mr Briggs began,—'My Lord, I have undertaken this long journey purposely to see your person, and to know by what engine of wit or ingenuity you came first to think of this most excellent help unto astronomy, viz. the Logarithms; but, my Lord, being by you found out, I wonder nobody else found it out before, when, now being known, it appears so easy.'"

Napier and Briggs must have been congenial spirits, for Briggs spent a month at Merchiston, returned for a second visit in the following summer, and intended to make a third visit in the next year; but Napier died before Briggs was free to set out for the north.

At the first visit Napier and Briggs discussed certain changes in the system of logarithms. In a letter to Napier before the first visit, Briggs had suggested that it would be more convenient, while the logarithm of the whole sine was still taken as zero, to take the logarithm of the tenth part of the sine as a power of 10, and he had actually begun the calculation of tables of his proposed system. Napier agreed that a change was desirable, and stated that he had formerly wished to make a change; but that he had preferred to publish the tables already prepared as he could not, on account of ill-health and for other weighty reasons, undertake the construction of new tables. He proposed, however, a somewhat different system from that suggested by Briggs, namely, that zero should be the logarithm, not of the whole sine but of unity, while, as Briggs suggested, the logarithm of the tenth part of the sine should be a power of 10. Briggs at once admitted that Napier's method was decidedly the better, and he set about the calculation of tables on the new system, which is essentially the system of logarithms now in use.

In an *Admonition* printed on the last page of some (not of all) of the copies of the *Canon Mirificus*, in a passage of the English translation, and in the dedication to the *Rabdologia*, Napier refers to the change of system, but does not state explicitly the share Briggs had in it, though in the dedication he speaks in the heartiest terms of "that most learned man, Henry Briggs, public professor of Geometry in London, my most beloved friend."

The copy of the *Canon Mirificus* in the Hunterian Museum of Glasgow University has the *Admonitio*. On the title-page is written, "Nathan Wrighte of Englefield." Can this be a relative of Edward Wright ?

Unfortunately, Dr Charles Hutton, in the excellent history of logarithms prefixed to the early editions of his Mathematical Tables, gives a version of this story that charges Napier with the intention of belittling Briggs's services, and of allowing no one but himself any credit in the improvement of the original system. It is very difficult to understand how Hutton could come to write as he did, especially as he has no justification in a single recorded word of Briggs himself. Briggs never to his last day spoke of Napier except in terms of the warmest affection, and never showed the least trace of a feeling that Napier had withheld from him any recognition such as Hutton demands. The friendship of Napier and Briggs was almost ideal in its sincerity and warmth, and Hutton's allegations are much to be regretted, occurring as they do in a work that has a deserved reputation for its general accuracy and wide knowledge of mathematical history.

The *Canon Mirificus* gave no account of the method by which the logarithms had been calculated. Napier there states that he prefers to show their use "that the use and profit of the thing being first conceived, the rest may please the more, being set forth hereafter, or else displease the less, being buried in silence. For I expect the judgment and censure of learned men hereupon, before the rest, rashly published, be exposed to the detraction of the envious." Napier did not himself publish any account of his method of calculating logarithms, but in 1619, after his death, his second work on logarithms, *Mirifici Logarithmorum Canonis Constructio*, came from the press of Andrew Hart under the editorship of Briggs, and with a preface by Robert Napier, the second son of his second marriage. In the Preface Robert Napier notes that in the book logarithms are called "artificial numbers" because his father "had this treatise beside him composed for several years before he invented the word Logarithms." It is interesting to observe the cordiality of his reference to Briggs ; "the whole burden of the business seems to have fallen on the shoulders of the most learned Briggs, as if it were his peculiar destiny to adorn this Sparta."

Robert Napier appears to have been his father's scientific executor ; among his papers was found a copy of a treatise by his father on Arithmetic and Algebra which bears the title, in Robert Napier's handwriting, "The Baron of Merchiston his Booke of Arithmeticke and Algebra. For Mr Henrie Briggs, Professor of Geometrie at Oxfoorde." Whether this treatise was ever sent to Briggs is not known ; it was edited for the Bannatyne Club by Mark Napier, and published in 1839 under the title *De Arte Logistica*.

"The whole burden" of calculating the new system of logarithms did

Photo by A. Swan Watson.

MERCHISTON CASTLE FROM THE WEST GARDEN.

By permission of George Smith, M.A., Headmaster of Merchiston Castle.

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in a very real sense fall on the shoulders of Briggs, and he devoted to the work conspicuous ability as well as unflagging zeal. Although this is not the occasion for an appreciation of Briggs, it may not be out of place to mention that the Simson collection in our University Library possesses a copy of Briggs's *Arithmetica Logarithmica* of 1624 with this inscription on the first blank leaf:—*Hunc mihi donavit Henricus Briggius Anno* 1625; on the title-page is the name Rob: Naper, and below this name is written in Simson's well-known hand

Rob: Simson, M.DCCXXXIII.

I do not know how this copy came into Simson's possession, but it is gratifying to have this tangible testimony to the good feeling that subsisted between Briggs and Napier's son.

The earliest publication of logarithms on the Continent was in 1617, when Benjamin Ursinus included in his *Cursus Mathematicus Practicus* Napier's canon of 1614, shortened two places. It was through this book that Kepler was aroused to the importance of Napier's discovery, though he had previously seen but not read the *Canon Mirificus*. His first hasty glance at the *Canon Mirificus* only led him to express himself in somewhat disparaging terms of the author—*Scotus Baro cujus nomen mihi excidit*; but when he had once studied the new method his enthusiasm was akin to that of Briggs. The dedication of his *Ephemeris* for 1620 consists of a letter to Napier dated from Lintz on the Danube, 28th July 1619. He was not aware that Napier had been then dead for more than two years. In the letter he states that he had used Napier's logarithms in the construction of this *Ephemeris* and therefore, of right, dedicated it to the "illustrious Baron." In 1624 Kepler published a table of Napierean logarithms with certain modifications and additions. It is perhaps worth noting that by comparing a letter from Kepler to Peter Crüger with a statement made by Anthony Wood in the *Athenæ Oxonienses* about a visit of Dr Craig, son of the feudist Sir Thomas Craig, one may reasonably infer that Napier was on the track of his logarithms as early as 1594.

This is not the occasion on which to pursue the history of Napier's logarithms. The credit of the invention is justly due to Napier and to Napier alone, but it would be very unjust to forget or to minimise the unique share that Briggs took in the promulgation of the logarithms. The tables in use at the present day are not those of Napier but those of Briggs, supplemented by those of Adrian Vlacq. Briggs seems to have been a man of the finest type, learned, able, and unselfish; it is only the merest justice to rank him alongside Napier in the history of the invention and calculation of logarithms.

Napier's conception of the logarithm cannot fail to suggest to the student of mathematics Newton's treatment of the fluxional calculus; not that Newton borrowed from Napier, but that the fundamental ideas of both are so much alike. The great generality of Napier's conception has been more clearly understood in recent years, and there is a strong tendency, at least so far as the advanced stages of mathematical study are concerned, to return to a definition of the logarithm that is equivalent to that of Napier.

In the course of his illustrations of the uses of logarithms Napier had frequent occasion to discuss trigonometric theorems, and the latest historian of trigonometry, Dr A. von Braunmühl, estimates Napier's work in this connection to be of the highest value. Napier, he considers, completely reorganised spherical trigonometry and enormously simplified the treatment of nearly all the trigonometrical formulæ.

Another achievement of Napier should be mentioned here, namely, that, though he did not introduce decimal fractions, he did introduce the decimal point, and showed, in the *Constructio*, the perfect simplicity and generality that attended its use. The decimal point is one of those simple devices that we take for granted but that needed a genius to invent; many years elapsed before its use became quite general.

In popular estimation it is perhaps the phrase *Napier's Bones* that most readily recalls his name. Though the device is now of little practical importance it is at least one more instance of Napier's faculty of combining simple practical applications with great theoretical insight. The bones are described, though not under that name, in the book :—*Rabdologiae, seu Numerationis per Virgulas Libri Duo : Cum Appendice de expeditissimo Multiplicationis Promptuario. Quibus accessit et Arithmeticae Localis Liber Unus.* (Edinburgh: Andrew Hart, 1617.) The book is dedicated to Alexander Seton, Earl of Dunfermline, and in the dedication Napier states that he was induced to publish a description of the construction and use of the "numbering rods" (that is, of the "bones") because many of his friends to whom he had shown them were so pleased with them that the rods were already almost common and were even being carried to foreign countries.

Mr Glaisher, in his article on Napier in the *Encyclopædia Britannica*, gives a clear account of the numbering rods or bones, and as I cannot improve upon it I transcribe it here. The bones as described by Mr Glaisher are slightly different from those that appear in the *Rabdologia*, but represent a common type.

The principle of "Napier's Bones" may be easily explained by imagining ten rectangular slips of cardboard, each divided into nine squares. In the top squares of the slips the ten digits are written, and each slip contains in its nine squares the first nine multiples of the digit which appears in the top square. With the exception of the top square, every square is divided into parts by a diagonal, the units being written on one side and the tens on the other, so that when a multiple consists of two figures they are separated by the diagonal. Fig. 1 shows the slips corresponding to the numbers 2, 0, 8, 5, placed side by side in contact with one another, and next to them is placed another slip containing, in squares without diagonals, the first nine digits. The slips thus placed in contact give the multiples of the number 2085, the digits in each parallelogram being added together; for example, corresponding to the number 6 on the right-hand slip we have 0, 8+3, 0+4, 2, 1; whence we find 0, 1, 5, 2, 1 as the digits, written backwards, of 6×2085. The use of the slips for the purpose of multiplication is now evident; thus, to multiply 2085 by 736 we take out in this manner the multiples corresponding to 6, 3, 7 and set down the digits as they are obtained, from right to left,

shifting them back one place and adding up the columns as in ordinary multiplication, viz., the figures as written down are

$$\begin{array}{r} 12510 \\ 6255 \\ 14595 \\ \hline 1534560 \end{array}$$

Napier's rods or bones consist of ten oblong pieces of wood or other material with square ends. Each of the four faces of each rod contains multiples of one of the nine digits, and is similar to one of the slips just described, the first rod containing the multiples of 0, 1, 9, 8, the second of 0, 2, 9, 7, the third of 0, 3, 9, 6, the fourth of 0, 4, 9, 5, the fifth of 1, 2, 8, 7, the sixth of 1, 3, 8, 6, the seventh of 1, 4, 8, 5, the eighth of 2, 3, 7, 6, the ninth of 2, 4, 7, 5, and the tenth of 3, 4, 6, 5. Each rod, therefore, contains

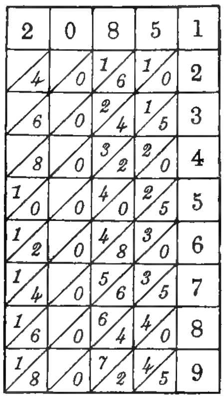

FIG. 1.

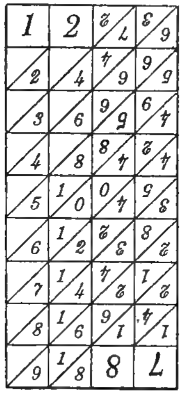

FIG. 2.

on two of its faces multiples of digits which are complementary to those on the other two faces; and the multiples of a digit and its complement are reversed in position. The arrangements of the numbers on the rods will be evident from fig. 2, which represents the four faces of the fifth bar. The set of ten rods is thus equivalent to four sets of slips as described above.

To the above extracts from Mr Glaisher's article I may add that the bones had a great vogue, and were very extensively used for several years after Napier's death. The *Rabdologia* was translated into Italian and Dutch, and the Latin edition was republished at Leyden. In *The Art of Numbring By Speaking-Rods: Vulgarly termed Nepeir's Bones*, which was published at London, in 1667, William Leybourn (who is denoted on the title-page simply as W. L.) gives a description of the rods, with examples of their use in multiplication, division, and the extraction of square and cube roots.

Sir Archibald Napier died in 1608; but before that date the relations between John Napier and the family of his father's second marriage had become very strained, and his succession to some of his father's estates was challenged. The dispute dragged on for some years, as Napier was not served heir of his father in the lands of Over-Merchiston till the 9th of June

1613. Other troubles emerged to disturb Napier's studies. The Raid of Glenfruin must have occupied his attention, and a curious document survives in which Napier on the one part and James Campbell of Laweris, Colin Campbell of Aberurquhill, and John Campbell of Ardnewnane, on the other part, make a bargain on the treatment to be meted out to any one " of the name of M'Grigour or any utheris heilland broken men " who may commit depredations on the Napier lands. The Campbells undertake " to use their exact diligence in causing search and try the committaris and doars of the said crymes," while Napier promises that he and his heirs will " fortifie and assist " the Campbells " in all their leasum and honest effairis, as occasioun sall offer."

Some writers have stated that Napier wasted his patrimony on his inventions, but there is no ground whatever for the statement. Napier knew very well how to look after himself, stuck tenaciously to what he held to be his rights (as in the family disputes over his succession; see also P. C. Reg., vi. 359), and handed down a very fine inheritance to his son Archibald, the first Lord Napier.

John Napier died on the 3rd of April 1617, and was buried in the church of St Cuthbert, Edinburgh. It is often stated that he was buried in St Giles, but it may now be held as established that it was in the church of St Cuthbert —the church in which he was an elder—that his body was laid.

David Hume (*History*, vol. vii. p. 44) casually refers to Napier as " the person to whom the title of a GREAT MAN is more justly due than to any other whom his country ever produced." Even though this judgment be challenged—and it is hard to decide who is most worthy of such an honourable title—every competent critic will concede that Napier's influence on the development of mathematics and its manifold applications in modern life was profound and far-reaching. A man of the highest culture, well versed in classical and theological learning, he was not exempt from the failings that are characteristic of the age in which he lived; in these he shows his kinship with common folk, and elicits our sympathy rather than our censure. But he was a man of pure and simple life, a sincere patriot, a genuine lover of spiritual religion and not merely an exponent of the particular forms it assumed in the confused theology of his day. In originality of conception and depth of insight he is one of the small band of mathematical thinkers, represented by Archimedes in antiquity and by Newton in more modern times, whose genius consolidated the labours of their predecessors and laid down the lines of future advance.

Though Napier's work was an essential condition of modern industrial development and reacted powerfully on modern thought, his name has little or no place in current text-books of Scottish history. Volumes have been written which record in minute detail the most petty squabbles of the sovereigns whose reign he adorned, but never mention his name. Yet he is known and honoured wherever modern science is taught, and he is a man whom every Scotsman should be proud to claim as a compatriot.

PORTRAIT OF JOHN NAPIER.

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Section B

LOAN COLLECTION, ANTIQUARIAN

I. Napier Relics.

1. (*a*) Set of "Napier's Bones" or Numbering Rods. Lent by Archibald Scott Napier, Esq. (*b*) Another set, lent by Miss Napier.
2. Collection of Books of John Napier of Merchiston. Lent by Archibald Scott Napier, Esq. See p. 30.
3. (*a*) Original Portrait of John Napier of Merchistoun; (*b*) Landscape of Merchistoun Castle. Lent by Miss Napier. (*c*) Portrait of John Napier. Lent by Sir A. L. Napier.
4. The Edinburgh University Portrait of John Napier of Merchiston, Inventor of Logarithms (1550–1617), is on view in the Senate Hall.
5. Napier Quadrant, from the Natural Philosophy Department, University of Edinburgh. Lent by Professor Charles G. Barkla, F.R.S.

Extract, by request of Professor James D. Forbes, from Town Council Records, by Mr Sinclair, August 1833, *vol. xiii. f.* 159:—

"Decimo Septimo Augusti Iajvj and Vigesimoprimo.

"The quhilk day the Proveist, baillies, dene of Gild, thesaurer and Counsall being convenit Ordains Peter Somivell, thesaurer, to give to Mr Johnne Hay to deliver to Ritchard Liver ressaver of the Customes of Londoun the soume of ten pundis sterling, and that for the pryce of ane grit quadrant ptening to him and sent hither to the Umqle Laird of Merchingstoun and whiche was delyverit be him to M. Andro Young Professor of the Mathematicks in King James' Colledge, and the same sal be allowit to him in his coptes and ordains the said instrument to be eikit to the Inventar of the Colledge and to be keipit to the use of the said Colledge and students thair."

6. Napier's Armchair. Lent by T. Blackwood Murray. This was bought by the great-grandfather of the present lender, at a sale of some of Napier's effects, at Merchiston Castle, early last century.
7. Merchiston Relics. Large China Bowl; Miniature of William, fourth son of John Napier; two Silver Salt Spoons; Seal. Lent by Miss Catherine Forrester of Stirling, a descendant of the above-named William Napier.
8. Bust of John Napier. Lent by Lord Napier and Ettrick.

II. Collection of "Napier's Bones" or "Numbering Rods."

(1) Lent by LEWIS EVANS, F.S.A.

English (2353). An early set of "Napier's Bones" of boxwood, 1620–1630, containing frame $3\frac{5}{8}$ ins. $\times 2\frac{1}{2}$ ins., nine bones $\frac{3}{16}$ in. square $\times 2\frac{1}{4}$ ins. long, and one for squares and cubes $\frac{7}{8}$ in. $\times \frac{3}{16}$ in. $\times 2\frac{1}{4}$ ins. The outer case of oak is more recent.

,, (2360). A complete set of "Napier's Bones" of boxwood, consisting of ten bones $\frac{1}{4}$ in. $\times \frac{1}{4}$ in. $\times 2\frac{1}{4}$ ins., and one for cubes and squares $\frac{9}{16}$ in. $\times \frac{1}{4}$ in. $\times 2\frac{1}{4}$ ins., in the original case $3\frac{7}{8}$ ins. $\times$ $2\frac{7}{8}$ ins. $\times \frac{3}{4}$ in. Ca. 1700.

,, (2359). A similar set bones, $\frac{7}{32}$ in. $\times \frac{7}{32}$ in. $\times 2\frac{1}{6}$ ins., squares and cubes $\frac{7}{32}$ in. $\times \frac{25}{32}$ in. $\times 2\frac{1}{16}$ ins., case $3\frac{5}{8}$ ins. $\times 2\frac{5}{8}$ ins. $\times \frac{5}{8}$ in., inscribed "Edmd. blow fecit—for Mr Julius Deedes 1715." 1715.

,, (2362). A complete set of "Napier's Bones" of bone in an ebony case. Bones $\frac{3}{16}$ in. $\times \frac{3}{16}$ in. $\times 2\frac{1}{16}$ ins., cubes and squares $\frac{5}{8}$ in. $\times \frac{3}{16}$ in. $\times 2\frac{1}{16}$ ins., case $3\frac{1}{2}$ ins. $\times 2\frac{11}{16}$ ins. $\times \frac{9}{16}$ in. Ca. 1700.

German (2370). A complete set of "Napier's Bones" of wood covered with printed paper, consisting of twelve bones with pyramidal ends at the base, in a hinged brass case $3\frac{1}{2}$ ins. $\times 2\frac{3}{8}$ ins. $\times$ $\frac{5}{16}$ in. The numbering runs upwards on these rods.

Flat Type

English (2368). An early set of "Napier's Bones" of the flat type, made of boxwood, and enclosed in a beechwood case having a boxwood frame for the bones on one side; the case contains four compartments, each capable of holding six bones. It has a lift-off lid. The case measures $5\frac{3}{4}$ ins. $\times 4\frac{1}{4}$ ins. $\times \frac{5}{8}$ in. The rods each measure $3\frac{3}{4}$ ins. $\times \frac{3}{4}$ in. $\times \frac{1}{8}$ in., the top of each being cut off at angle of 45°. Only nine now remain. This type is described in *The Art of Numbering by Speaking-Rods*. W. L. (Wm. Leybourn). 16mo. London, 1667. Ca. 1680.

,, (2369). A set of "Napier's Bones" of another type—flat—made of boxwood, in a case or "tabulet" 6 ins. $\times 2\frac{15}{16}$ ins. $\times \frac{9}{16}$ in.; at each end of the case is a bevelled slope with index numbers, running upwards 1 to 9. In these slopes are four holes, probably for containing pins to "prick off" the part divided in sums of division. Marked on the upper edge of the "tabulet" is "Divisor" to the left, and "Multiplicand" to the right, with T (top or total) in the centre; the lower edge has on it—from right to left—I, X, C, M, X, C, MM, X, C, M, X, C, representing units,

tens, etc., up to 100,000,000,000, and between each of these numbers is a hole for the "pricking-off" pins. At the bottom of the case is a printed paper of instructions for division and multiplication by "The Fore Rule" and "The Backe Rule." There are now nineteen bones in the case, $1\frac{3}{4}$ ins. $\times \frac{7}{16}$ in. $\times \frac{1}{16}$ in. full; each has a curved nick in its upper end to facilitate removal from the case. Probably there were originally twenty ordinary bones in the set and one twice as wide for squares and cubes. The lid or cover is missing. Ca. 1700.

English (2366). A set of cylindrical "Napier's Bones" in a box $4\frac{3}{4}$ ins. $\times$ $2\frac{1}{2}$ ins. $\times 1\frac{1}{16}$ ins., all made of boxwood.

Outside the hinged lid is a table giving the interest at 6 per cent. for one, two, three, six, and twelve months, for each £10 from 10 to 90, and for hundreds of pounds to £500.

On the bottom are two tables. The first shows the year, w.d. (week day), epact from (16)79 to 93, and a "Perpetual Almanack" with the year beginning in March.

The second table shows the time of the tides at various places, in relation to the moon's age.

Inside the lid is an addition table (?) in thirteen columns of eleven numbers, the first numbered downwards from 0 to 10, the next from 1 to 11, and so on, the thirteenth from 12 to 22.

The body of the instrument contains six boxwood cylinders, each $\frac{5}{8}$ in. diameter and $1\frac{13}{16}$ ins. long; each of these cylinders has marked on it columns of the digits 1 to 9 multiplied by the numbers 0 to 9; by means of thumbscrews projecting through the front of the case, these cylinders can be turned so that any desired series of multiples may be uppermost, and thus serve as the ordinary "Napier's Bones." The wooden coverplate with the nine digits at one end and their cubes at the other is a restoration. 1679.

(2) Lent by ANGUS M. GREGORSON, W.S.

These belonged to the Rev. Colin Campbell, M.A., Minister of Ardchattan Parish, Argyllshire, from 1667 to 1726. Born 1644, died 1726. He was an astronomer and mathematician and corresponded with Newton, James and David Gregorie, Maclaurin, and Leibnitz. Newton, in a letter to Professor Gregorie, is reported to have said of him: "I see that if he were among us he would make children of us all." See *Dictionary of National Biography* on Rev. Colin Campbell.

(3) Lent by W. J. MERCER DUNLOP, Esq.

(4) Lent by John Robb.

This set of rods differs in some respects from those usually found. The ten rods of which it consists are not enclosed in a box, but are threaded on to a stout straight steel wire, from which they can be readily detached as required. Each rod is about five inches long and of square cross-section, the side of the square being about four-tenths of an inch.

Each rod shows in the usual way the multiples of the numbers 0, 1, 2 . . . 9, but the numbers on each rod run from top to bottom on each of the four sides, so that a rod has never to be reversed as is usual in *Napier's original* form.

These rods are stamped with the date 1803, and belonged to Mr William Harvey, who in conjunction with Mr Jackson explored Australia. They are now the property of Mr John Robb, Glasgow, who inherited them through his mother, Mr Harvey's cousin.

III. Facsimiles of the Title-Pages of the Editions of the Works of John Napier of Merchiston. Lent by William Rae Macdonald, F.F.A.

IV. Portable Sundials. By John R. Findlay, M.A., D.L.

Portable sundials are of more recent invention than fixed dials, though examples have been found dating from the early Roman Empire. They continued in use till the beginning of the nineteenth century, and were made in very considerable numbers in France, Germany, Italy, and England from the beginning of the sixteenth to the middle of the eighteenth century. The majority of those in the collection exhibited are seventeenth-century dials; the earliest date from the middle of the sixteenth century, and the latest is dated 1801.

Portable sundials are necessarily more elaborate than fixed dials, since fixed dials are constructed for a given latitude, and levelled and set in the meridian once and for all, when erected. If portable dials are to be of general use, some means of adjustment for latitude must be provided. In using them, it was presupposed that the latitude was known, and in most cases they have engraved on them the latitudes of a number of important towns. Given the latitude, it is also necessary to fix the zenith. This is done either by hanging up the instrument or by levelling it by means of a plummet. Given the latitude and zenith, it is possible to find the hour either by the altitude or azimuth of the sun. If the altitude is chosen, it is necessary to know the sun's declination for the day of the year on which the

dial is used. If the azimuth is chosen, the instrument must provide some means of finding the meridian. This is done by a compass, or by the combination of two dials of different construction. If both dials show the same hour, then the instrument is on the meridian. Of the two types the altitude instruments were perhaps the simpler, but they have the disadvantage that it is impossible to discriminate between the time before noon and after noon, and that for the period in the middle of the day the changes in the quantity measured are smallest. Both types were used concurrently, though the azimuth instruments were by far the commoner. The earlier compass dials were constructed for a magnetic variation corresponding to the date at which they were made, and the angle of variation allowed for provides a means of determining their date. Before 1660 the variation in Europe was east; after that date it was west. Between 1500 and 1700 it was much the same for the whole of Europe; after that date it began to vary in different localities, and some of the more modern instruments provide a means for adjusting them for different variations.

Of dials which determine the hour by the sun's altitude there are three main types. The simplest of these is the ring dial, in which the hours are represented by graduations on a circle, and a spot of light falling through a small hole, the position of which is fixed according to the sun's declination, gives the time. Analogous to these is the multiple-ring dial or armillary sphere, consisting of three concentric rings or two concentric rings and a cross bar fixed with a slide. On this slide or on one of the rings is a sight which can be adjusted to the sun's declination. In the more elaborate examples positions for a cycle of four years are shown. By means of one of the rings the instrument can be adjusted for various latitudes. It is then hung up and swung round till the sun's image formed by the small hole falls on the scale. In the second type, there is a horizontal gnomon, and the hour lines are curves of the length of the shadow of this gnomon on a vertical surface, according to the hour and season. These curves are either drawn on a cylinder, as in the pillar or "shepherd's" dial, or on a flat surface. The seasons are represented by vertical lines. The gnomon is moved to the appropriate line, the dial placed so that its shadow falls perpendicularly, and the hour is read on the hour line. This type of dial was not "universal," as it was always made for a given latitude.

In the third type the altitude was found by means of sights, and the hour was read by a sliding bead on a plumb line, the bead being "rectified" or set according to the declination. A great deal of ingenuity was displayed in the devising and construction of these dials, and they took various forms. The simplest of these is the quadrant, though it was often complicated by the addition of other lines and constants. Some of these dials can be adjusted for any latitude, the point of suspension of the plumb line being determined by what was known as a "trigon" of signs and latitudes.

The azimuth dials resemble more closely the ordinary types of fixed dial. They have either a string as a gnomon or one of the ordinary type, the dial being generally horizontal. The adjustment for latitude takes various forms. In the majority, especially in the case of a number of French dials which were made in Paris in the end of the seventeenth century, the angle of the gnomon

can be adjusted by a quadrant scale, two, three, or four dials projected for various latitudes being engraved on the dial plate. In others the dial plate can be tilted. In the case of the ivory dials with a string gnomon, different holes were provided for different latitudes. These ivory dials had often vertical equatorial and equinoctial dials engraved on them as well.

Equatorial dials, in which the scale is on a circle set in the plane of the equator, and adjusted for different latitudes by means of a quadrant graduated in degrees, are a distinct class. Owing to its simplicity of construction it became a favourite type. A large number were made in Germany in the end of the seventeenth century, and the later French dials are generally of this form.

Another type is the analemmatic dial, in which there are two dials, one an ordinary one with a sloping gnomon, the other with a sliding upright gnomon and a dial founded on the projection of the sphere known as the analemma. In this case there is no compass, but the meridian is found by turning the instrument round till both dials show the same hour. In another example the meridian is found by the projection of curves giving the position of the shadow of a notch on the gnomon at the beginning of each month.

In most cases the hour given is the ordinary astronomical hour, but in the earlier dials there are subsidiary dials by which the Italian hours, reckoned from sunset to sunrise, or the Babylonian hours, reckoned from sunrise to sunset, can be determined. One example has also a graduated pointer and scale by which the Jewish, " planetary," or unequal hours can be determined; the periods from sunrise to sunset and from sunset to sunrise being each divided into twelve equal hours, these, of course, being of different lengths in summer and winter.

In some cases a lunar dial, by which the hour of the night can be determined by the shadow given by the moon, is added, and sometimes a nocturnal which gives the hour of the night by a simple observation of the Pole Star and a fixed star—Kochab, in the Great Bear, being generally selected.

During the period that these dials were in use the ordinary time was solar time, and watches and clocks were set to it. On dials made after mean solar time came into use, a table of corrections is generally to be found; while after the reform of the calendar, none of the English altitude dials, on which the equinox is always marked at 10th March, would be of any use.

Most of the types shown in the collection will be found described in Bion's treatise on the construction and use of mathematical instruments.

Catalogue of Portable Sundials

Folding Azimuth Dials with String Gnomon

1. Copper gilt, compass and plummet, table of length of day and entry of sun in signs of zodiac. V.S. 1584, German.

2. Ivory, mounted with brass gilt, two compasses, one with three horizontal string dials, other with points of compass, vertical dials for Italian hours and length of day, lunar dial. German, *circa* 1620.

3. Copper gilt, compass, adjustment with spring drum for four latitudes,

lunar dial, calendar of planetary hours and graduated eccentric and pointer for finding planetary hours. French, sixteenth century.

4. Ivory, compass, horizontal string dial for five latitudes, gnomon dial for length of day, pin dials "weisch uhr" and "grose uhr," lunar dial with points of compass and winds. German, 1649.

5. Copper gilt, dial plate restored. Arms, Cor. Drebbel, 1579, German.

7. Brass, compass, folding support for plummet, adjustment for deviation. English, eighteenth century.

8. Ivory, compass, three horizontal string dials, vertical dials for length of days and Italian hours, two horizontal cup dials, points of compass, lunar dials for Julian and Gregorian epacts, compass card, serpent mark of T. Ducher. German, *circa* 1625.

9. Ivory, compass, vertical and horizontal string dial, gnomon dial for length of day, pin dials for Italian and Babylonian hours, lunar dial and points of compass. German, Lienhart Miller, 1605.

10. Ivory (small), compass, string horizontal dial, equatorial and equinoctial dials for different latitudes. French, seventeenth century.

11. Ivory, compass, horizontal string dial, gnomon dial for length of day, pin dial for Italian hours, lunar dial and points of compass. German, Lienhart Miller, 1619.

12. Ivory, compass, vertical and horizontal string dial, pin Italian dial, lunar dial. T. D. and Dragon. Nien Perger. German, seventeenth century.

13. Ivory, compass, string horizontal dial, equatorial and equinoctial dial with adjustment for various latitudes, sliding analemma, lunar dial and calendar. French, seventeenth century.

14. Wood, compass, vertical and horizontal printed and coloured paper. David Beringer. French, eighteenth century.

15. Copper gilt and tinned, compass, level and cord, lunar dial. Johann Martin, Augsburg. Early eighteenth century.

75. Ivory, compass, horizontal string dial for five latitudes, pin dials for length of day, Italian and Babylonian hours, lunar dial, compass card and winds. Leonhart Muller, 1637. German.

76. Ivory, compass, horizontal string dial for four latitudes, pin dials for Italian hours and length of day, lunar dial. Hans Ducher, Nuremberg, 1580.

16. Bronze, Japanese, to show noon only.

Azimuth Dials with Solid Gnomon

19. Silver, and black enamel partly gilt, octagonal compass, three dials. Chapotot à Paris. French, late seventeenth century.

20. Silver, .compass, octagonal, three dials. Sautout l'aine à Paris. French, late seventeenth century.

21. Silver, compass (small size), four dials. Butterfield à Paris. French, late seventeenth century.

26. Silver, compass, octagonal, four scales. Butterfield à Paris. French, late seventeenth century.

28. Brass, large compass, octagonal, tilting plate, graduated compass. Chapotot à Paris. French, *circa* 1700.

29. Copper gilt, square, on legs, chased and pierced for latitude. French? sixteenth century.

30. Brass box, three tiers over compass. French, *circa* 1700.

31. Brass, square, with pierced gnomon. Bartholomew Newsum. English, sixteenth century.

32. Copper tinned, single dial. Leo Hay, Bamberg. German, early eighteenth century.

33. Brass, dial over compass. English, eighteenth century.

34. Brass, square wooden base, dial over compass. French, *circa* 1700.

35. Copper gilt, levelling screws and plummet and curves of altitude. Fecit Joan Engelbrecht, Beraunensis in Bohemia. German, late eighteenth century.

36. Wooden, circular dial on compass card. German.

37. Copper gilt, box dial with moving dial plate, nocturnal, lunar dial and compass. German, 1587.

67. Cube on pillar, with five dials, printed paper compass. David Beringer. French, *circa* 1780.

74. Brass, octagonal, with tilting dial plate adjustment for variation of compass. Jacques le Maire au Génie, Paris. French, *circa* 1700.

Equatorial Azimuth Dials

38. Brass, mounted on stand, with levelling screws and plummet, screw adjustment for altitude on slide, and scale for four years.

41. Brass, compass, octagonal, engraved, adjustment for deviation.

42. Copper gilt, with two semicircular dials, adjustment for declination. Johan Muller in Augsburg. Seventeenth century.

43. Brass (large), compass, octagonal, needle to set for deviation. Clerget à Paris au Butterfield. French, eighteenth century.

47. Silver, compass, octagonal, large levelling screws, compass stop, graduated scale for deviation. Secretan à Paris. Eighteenth century.

48. Copper gilt and tinned, compass, plummet missing. Johan Willebrand in Augsburg. Late seventeenth century.

49. Brass, compass, square, engraved, with plummet. L. Grassl. German, eighteenth century.

50. Copper gilt, compass, string gnomon, plummet, levelling screws, cog wheel with pointers to show hours and minutes. German, *circa* 1750.

61. Brass, octagonal. T. Nholdernich. German, *circa* 1750.

Armillary Dials

51. Brass, with slide. J. Coggs fecit. English, seventeenth century.

52. Copper gilt and tinned, with sliding scale for four years. French?

53. Copper gilt. Johan Somer, Augsburg.

54. Brass, three ring. French?

55. Brass, tinned on stand levels and levelling screws, graduated compass, cog-wheel motion, sights for use as theodolite.

56. Brass, 8-inch, slide and graduations for determining sun's altitude. English, early eighteenth century.

Analemmatic Dials

57. Brass, folding, with perpetual calendar for Sunday, letters. Thos. Tuttall, Charing Cross. English, 1697.

58. Brass, pierced, single gnomon, tilting and sliding plate. Johanathan Sisson. English, 1735.

59. Copper tinned, with "furniture." Joan Engelbrecht, Beraunensis. German, 1801.

Miscellaneous

62. Copper gilt, heart-shaped, ivory scale and gnomon on compass scale.

69. Dial on spoon, with small compass, copper gilt. Augsburg. Sixteenth century.

70. Dial, in form of crucifix. French, seventeenth century.

Azimuth Dials

63. Copper gilt, cover for ivory tablets, pin gnomon. German, seventeenth century.

64. Copper gilt, circular dial with sliding gnomon and nocturnal on back. German, seventeenth century.

71. Copper, silvered "Monk's Head" dial one side, "Trigon of Signs" universal dial other side. French, seventeenth century.

72. Brass, nocturnal and lunar dial one side, "Trigon of Signs" on other. Caspar Vogel, Cologne, 1541.

73. Copper gilt, nocturnal and small gnomon compass dial other side. French, sixteenth century.

68. Wooden, "shepherd's" dial. French, seventeenth century.

60. Brass, quadrant, hours, azimuth, and other constants for lat. 57. H. Sutton. English, 1657.

61. Copper gilt, quadrant, for lat. 45, Italian hours, reverse on back. Italian, sixteenth century.

Other Instruments

Astrolabe, Bronze.—French. Fifteenth century, with four tables for different latitudes.

Astrolabe, Brass.—Italian. Early sixteenth century, with two tables, engraved both sides for different latitudes.

Theodolite.—French. Late seventeenth century, with compass and sights, made by Butterfield à Paris.

Dialling instrument, with compass, brass gilt. German, seventeenth century.

Dialling instrument or theodolite, bronze, with sights. Italian, sixteenth century.

Sector.—Brass. French, seventeenth century.

Sector.—French, late seventeenth century. Butterfield à Paris.

V. Photographs of Calculating Machines exhibited in the Science Museum, South Kensington. Lent by the Board of Education.[1]

Frame 1.—A. Napier's Rods; seventeenth century.

The rods are strips of boxwood, and some are shown arranged for the multiplication of any number by 765479.

B. Napier's Rods; Italian, seventeenth century.

The rods are of brass and square in section, with numbers on each face.

C. Napier's Rods, p. 3, and p. 6; cylindrical.

The numbers are arranged on rollers contained in a case.

D. Title-page of book in which the device is first described. *Rabdologiæ*. Edinburgh, 1617.

E. Portrait of John Napier, Baron of Merchiston. Born 1550, died 1617.

Frame 2.—A. Morland's Calculating Machine.

Invented by Sir Samuel Morland and made in 1666, for the mechanical addition of sums of British money.

B. Similar to A.

In this example the top plate is shown removed, to exhibit the internal arrangement.

C. Morland's Trigonometrical Machine.

Made by Sutton and Knibb in 1664.

D. Title-page and next page of book in which the machine is first described. *The Description and Use of Two Arithmetick Instruments*. London, 1673.

E. Portrait of Sir Samuel Morland. Born 1625, died 1695.

Frame 3.—A. Stanhope's Calculating Machine, 1775.

This was made by Jas. Bullock in 1775 for Viscount Mahon, afterwards third Earl Stanhope.

Multiplication is performed by repeated addition, and division by repeated subtraction.

A complete cycle (corresponding with one turn of the handle of a Thomas de Colmar Arithmometer) is effected by moving the sliding rectangular frame to and from the operator for multiplication, and from and to the operator for division.

B. Stanhope Calculating Machine, 1777.

By the same maker as the above.

The "to-and-fro" motion of the 1775 machine is here replaced by rotation. A complete cycle is effected by one turn of the handle, clockwise for multiplication, and anti-clockwise for division.

[1] Copies of the photographic prints or lantern slides of these instruments may be obtained at the Science Museum, South Kensington, London, S.W.

C. The same machine as B.
The top plate is displaced so as to show the internal construction.

D. Portrait of Viscount Mahon, afterwards third Earl Stanhope. Born 1753, died 1816.

Frame 4.—A. Babbage's Difference Engine.
This shows a small portion of the machine invented by Charles Babbage for calculating and printing tables of numbers.
The construction of the machine was commenced in 1823 by authority and at the cost of the Government, the work being suspended in 1833, and abandoned by the Government in 1842.
The whole engine, when completed, was intended to have had twenty places of figures and six orders of differences.

B. Babbage's Analytical Engine.
This shows a portion of the analytical engine commenced in 1834 by Charles Babbage, with the object of calculating and printing the numerical value or values of any function of which the mathematician can indicate the method of solution. (See *Babbage's Calculating Engines.* Published by E. & F. N. Spon, London.)

C. Babbage's Analytical Engine.
This shows the "mill" of the analytical engine as put together by Major-General H. P. Babbage, the youngest son of the inventor.

D. Portrait of Charles Babbage. Born 1791, died 1871.

Frame 5.—A. Scheutz's Difference Engine.
This shows the difference engine made in 1859 by Bryan Donkin under the direction of the inventor.
The machine was used by Dr Farr at Somerset House for computing and printing portions of the English Life Table. (See *Tables of Lifetimes, Annuities, and Premiums.* With an Introduction by William Farr, M.D., F.R.S., D.C.L. Published by Longman, Roberts & Green. London, 1864.)

B. Detail of printing mechanism.

C. Detail of figure wheels and mechanism for the operation of "carriage," etc.

D. Impression on card printed by the machine. From this a stereotype is prepared for printing purposes.

E. Impression from stereotype.

F. Portrait of the inventor.

VI. Letters from Scottish Mathematicians to the Rev. Colin Campbell, M.A., Minister of Ardchattan, Argyllshire, 1667–1726. Lent by ANGUS M. GREGORSON, W.S.

(1) Five letters from Professor James Gregorie to the Rev. Colin Campbell, Minister of Ardchattan, Argyll, from 1667 to 1726.
(2) Sixteen letters from Professor David Gregorie to the same.
(3) Three letters on mathematical subjects, etc., by Professor James Gregory, Edinburgh, to the same.
(4) Three letters from Colin Maclaurin. A letter from his uncle, sending Mr Campbell a "Double" of a Thesis by Colin Maclaurin, then at Glasgow College (aged thirteen).
(5) Ten letters from J. Craig, 1687–1708.
(6) Seven letters from Dr Pitcairn, 1703–1710.
(7) Letter from Dr George Cheyne.
(8) Letter from Robert Simson, Professor of Mathematics, Glasgow, 1717.
(9) A description in manuscript of "Dr Godfredius . . . Leibnitz, his watch," and on same paper "The Description of Hugon his watch."
A drawing of Mr Hugon his watch, on separate paper.

VII. Davis Quadrant. Lent by the Rev. A. HORSBURGH, M.A.

THIS naval quadrant was invented by the great Arctic navigator John Davis, the discoverer of Davis Straits.

It remained the standard instrument in the Navy from the days of Napier till as late as the time of Anson, who used one such as this on his memorable voyage round the world. Rough as the instrument appears, it marked a great improvement on the cross-staff. The observer turned his back to the sun and shifted the vane on the smaller arc till it cast a sharp shadow on the horizon slot. A diagram is attached to the instrument showing how it was used.

VIII. Exhibits by Adam Henderson, F.S.A.

(i) An Arithmetical pastime, intended to infuse the rudiments of Arithmetic, under the idea of amusement.

Oblong sheet, mounted on cloth ; size, when unfolded, $12\frac{1}{2}$ inches $\times$ $27\frac{7}{8}$ inches ; undated, probably *c.* 1810.

(ii) Macfarlane's Calculating Cylinder. "The machine consists of a small cylinder, having three distinct parts revolving separately, on which are

inscribed several series of numbers, calculated to propose and answer questions to an almost indefinite extent in the first four rules of Arithmetic: in the rules of Reduction, Proportion, Practice, and Interest."

(iii) Rules, directions, and examples, illustrating the use of Macfarlane's Calculating Cylinder, designed to promote the instruction of youth in the elementary principles of Arithmetic, adapted to public and private tuition. By James Macfarlane, teacher of the Mercantile Academy, George Square, Glasgow.

12mo, cloth. Glasgow, 1833.

(iv) A Secular Diary for ascertaining any day of the week or month in either the old or new style, commencing 1601, and continued up to the year 1900. By D. Barstow.

Sheet, mounted on cloth, and pasted into a cloth case; size, when unfolded, $14\frac{1}{4}$ inches × 11 inches. Date, July 5, 1836.

Section C

MATHEMATICAL TABLES

I. Catalogue of Historical Books exhibited at the Napier Tercentenary Celebration, 1914. By Professor Sampson, F.R.S.

(The detailed description will be published in the Memorial Volume.)

The books catalogued below may be classified in the following divisions :—

1. Napier's work on the Apocalyse, in its various editions.
2. The editions of the *Description* and *Construction* of Logarithms.
3. The editions of the *Rabdology*.
4. *De Arte Logistica*.
5. The calculations of Briggs, Gunter, Vlacq, Kepler, and Ursinus.
6. References to the co-discovery of logarithms by Jobst Bürgi.
7. The *Opus Palatinum* of Rheticus, with the additions of Pitiscus, the great table of natural sines, etc., preceding Napier's discovery.
8. References to the method of *Prosthaphæresis*, an earlier alternative for facilitating multiplications.
9. Specimens illustrating the subsequent history of logarithmic tables.

In preparing this collection much use has been made of Dr J. W. L. Glaisher's admirable article on "Logarithms" in the *Encyclopædia Britannica*, and of the bibliographical descriptions in W. R. Macdonald's catalogue appended to his English version of the *Construction* published in 1889.

The Society is indebted for the loan of these volumes to Alexander Scott Napier, Esq.; L. Evans, Esq., University College, London; W. R. Macdonald, Esq., Edinburgh; Dr Hay Fleming, Edinburgh; John Spencer, Esq., London; the Universities of Edinburgh and Glasgow; the Royal Observatory, Edinburgh (Crawford Library); and the Town Library, Dantzig.

I. Napier's Work on the Apocalypse

1. A Plaine Discovery of the whole Revelation of Saint John. . . . Edinburgh. Printed by Robert Waldegrave. 1593. 8vo, size $7\frac{1}{4} \times 5$ inches.

Lent by W. R. Macdonald, Esq.

1A. The same—Newlie Imprinted and Corrected. London. Printed for John Norton. 1594. 8vo, size 7×5 inches.

Lent by Dr Hay Fleming.

2. Ovverture de tous les Secrets de l'Apocalypse ou Revelation de S. Jean. . . . Par Jean Napier, (c.a.d.) Nonpareil Sieur de Merchiston. A La Rochelle, 1602. 4to, size $6 \times 8\frac{1}{2}$ inches.

Lent by Archibald Scott Napier, Esq.

2A. Ouverture de tous les Secrets de l'Apocalypse ou Revelation de S. Jean. . . . Par Jean Napeir, (c.a.d. Nonpareil) Sieur de Merchiston. A La Rochelle, 1607. 8vo, size $4\frac{1}{2} \times 5\frac{1}{4}$ inches.

Lent by Archibald Scott Napier, Esq.

3. Een duydelijcke verclaringhe Van de gantse Open-baringhe Joannis des Apostels. . . . Wt-ghegheven by Johan Napeir, Heere van Marchistoun. Middelburch, 1607. 8vo, size $4\frac{1}{2} \times 6\frac{1}{2}$ inches.

Lent by Archibald Scott Napier, Esq.

4. Johannis Napeiri, Herren zu Merchiston. Eines trefflichen Schottlandischen Theologi, schön und lang gewünscht. Auslegung der Offenbarung Johannis, . . . Getruckt zu Franckfort, 1615. 8vo, size 4×7 inches.

Lent by Archibald Scott Napier, Esq.

5. Napier's Narration: or, An Epitome of his Booke on the Revelation. . . . London, printed for Giles Calvert, 1641. 4to, size $5 \times 6\frac{3}{4}$ inches.

Lent by Archibald Scott Napier, Esq.

II. Editions of the Description and Construction of Logarithms

6. Mirifici Logarithmorum Canonis descriptio. Authore ac Inventore Joanne Nepero, Barone Merchistonii, etc., Scoto. Edinburgi, 1614. 4to, size $7\frac{1}{2} \times 5\frac{1}{2}$ inches.

Lent by the Royal Observatory.

7. The same as 6, except that the last page (m 2) contains the *Admonitio* expressing an intention of publishing later an improved form of logarithms. 1616.

Lent by the University of Edinburgh.

8. Mirifici Logarithmorum Canonis Descriptio (Title-page only). Mirifici Logarithmorum canonis Constructio. Edinburgi, 1619. 4to, size $7\frac{3}{4} \times 6$ inches.

Lent by the Royal Observatory.

9. Logarithmorum canonis Descriptio. Authore ac Inventore Joanne Nepero. In the same volume: Mirifici Logarithmorum canonis Constructio. Lugduni, 1620. 4to, size $5\frac{1}{2} \times 7\frac{1}{4}$ inches.

Lent by Archibald Scott Napier, Esq.

10. Logarithmorum Canonis Descriptio . . .: Lugduni, 1620. Mirifici Logarithmorum Canonis Constructio: Lugduni, 1620. Nearly identical with the foregoing.

Lent by Archibald Scott Napier, Esq.

11. A Description of the Admirable Table of Logarithms. Invented and published in Latin by that Honorable John Nepair, Baron of Marchiston, and translated into English by Edward Wright. London, 1616. 12mo, size $5\frac{3}{4} \times 3\frac{1}{4}$ inches.

Lent by the Royal Observatory.

12. A Description of the Admirable Table of Logarithmes. Translated into English by Edward Wright. With an addition of the Instrumentall Table described in the end of the Booke by Henrie Brigs. London, 1618. The book is the same as the foregoing, with a slight change and addition to the title, and, corresponding to it, following Briggs's account of proportional parts, eight pages containing "An Appendix to the Logarithms, showing the practise of the Calculation of Triangles."

Lent by the University of Edinburgh.

13. The Construction of the Wonderful Canon of Logarithms by John Napier, Baron of Merchiston. Translated from Latin into English, with Notes and a Catalogue of the various editions of Napier's Works, by William Rae Macdonald, F.F.A. William Blackwood & Sons, Edinburgh and London, 1889. 4to, size 8×10 inches. This is one of the most important works on Napier.

Lent by the Royal Observatory.

III. Editions of the Rabdology

14. Rabdologiæ, sev Numerationis per Virgulas libri dvo: cum Appendice de expeditissimo Multiplicationis Promptuario. Authore & Inventore Joanne Nepero, Barone Merchistonii, etc., Scoto. Edinburgi, 1617. 12mo, size $5\frac{5}{8} \times 3\frac{1}{4}$ inches.

Lent by the Royal Observatory.

15. Rabdologiæ, sev Numerationis per Virgules libri duo: cum Appendice de expeditissimo Multiplicationis Promptuario. . . . Authore & Inventore Joanne Nepero, Barone Merchistonii, etc., Scoto. Luduni, 1622. 12mo, size $3\frac{1}{8} \times 5\frac{3}{8}$ inches.

Lent by Archibald Scott Napier, Esq.

16. Raddologia, Overo Arimmetica Virgolare in due libri diuisa: con appresso un' espeditissimo Prontvario della Molteplicatione. . . . Auttore & Inventore il Barone Giovanni Nepero. Tradottore dalla Latina nella Toscana lingua il Cavalier Marco Locatello. . . . Verona, 1623. 8vo, size $6\frac{1}{8} \times 4\frac{1}{8}$ inches.

Lent by the Royal Observatory.

17. Rhabdologia Neperiana—a German translation of Book I. by M. Benjaminem Ursinum. Berlin, 1640. 4to, size $5\frac{1}{2} \times 6$ inches.

Lent by University College, London.

IV. De Arte Logistica

18. De Arte Logistica, Joannis Naperi. Edinburgi, 1839. 4to, size $8\frac{1}{4} \times 10\frac{1}{2}$ inches.

Lent by the Royal Observatory.

18a. Another of the same, with original MS.

Lent by John Spencer, Esq.

V. The Calculations of Briggs, Gunter, Vlacq, Kepler, and Ursinus

19. Canon Triangulorum, or Tables of Artificiall Sines and Tangents to a Radius of 100,000,000 parts, . . by Edward Gunter. London, 1636.

Lent by J. Ritchie Findlay, Esq.

20. Benjaminis Ursini Mathematici Electoralis Brandenburgici Trigonometria cum magno Logarithmor. Canone . . . 1625. 4to, size $5\frac{1}{2} \times 7\frac{1}{2}$ inches. Coloniæ.

Lent by the University of Edinburgh and by W. Rae Macdonald.

21. Arithmetica Logarithmica sive Logarithmorum Chiliades Triginta. . . . Hos Numeros primus inventit Clarissimus Vir Johannes Neperos . . . et usum illustravit Henricus Briggius. . . . Londoni, 1624. 4to, size $12 \times 7\frac{3}{4}$ inches.

Lent by the Royal Observatory.

22. Another copy of the same, with inscription on the front flyleaf, " Hunc mihi donavit Henricus Briggius anno 1625 " ; and on the title-page, " Rob. Naper " and " Rob : Simson, M·DCCXXXIII."

Lent by the University of Glasgow.

23. Arithmetica Logarithmica, sive Logarithmorum Chiliades Centum. . . . Hos Numeros primus inventit Clarissimus Vir Johannes Neperos . . . et usum illustravit Henricus Briggius, in celeberrima Prof. Savilianus. Editio Secunda per Adrianum Vlacq Gondanum. Goudæ, 1628. 6mo, size $13 \times 8\frac{1}{2}$ inches.

Lent by the Royal Observatory.

24. Arithmetique Logarithmetique . . . par Jean Neper . . . change par Henry Brigs, et traduite du Latin en Francois par A. Vlacq. Goude, 1628.

Lent by Professor R. A. Sampson.

25. Henrici Briggii, Tafel van Logarithmi. . . . Goude, 1626. 8vo, size $7 \times 4\frac{1}{2}$ inches.

Lent by the Royal Observatory.

26. Nievwe Talkoust in hovende de Logarithmi, . . . ghemæcht van Henrico Briggio (with tables of log. sines and tangents) ghemacht van Edmund. Guntero. Goude, 1626. 8vo, size $7\frac{3}{4} \times 4\frac{3}{4}$ inches.

Lent by University College, London.

27. Logarithmicall Arithmetike, or Tables of Logarithms for Absolute Numbers . . first invented by John Napier . . . transformed by Henry Briggs, and Sir Henry Savils. London, 1631. 4to and 6mo.

Lent by the Royal Observatory.

28. Trigonometria Britannia, sive De Doctrina Triangulorum libri duo. Henrico Briggio. Goudæ, 1633. 4to, size $13 \times 8\frac{1}{2}$ inches.

Lent by the Royal Observatory.

29. Trigonometria Artificialis: sive Magnvs Canon Triangulorum Logarithmicvs . . . Henrici Briggii. Goudæ, 1633. 4to, size $13 \times 8\frac{1}{2}$ inches.

Lent by the Royal Observatory.

30. Joannis kepleri . . . Mathematici Chilias Logarithmorum. Marpurgi, 1624. 4to, size $7\frac{1}{2} \times 6\frac{1}{2}$ inches. Joannis Kepleri . . Mathematici supplementum Chiliadis Logarithmorum. . . . Marpurgi, 1625. 4to, same size.

Lent by the Royal Observatory.

31. Tabulæ Rudolphinæ . . . Joannes Keplerus. Ulmæ, 1627. 4to, size 9×13 inches.

VI. The Discovery of Logarithms by Jobst Buergi

32. Buergi, Arithmetische und Geometrische Progress Tabulen. . . . Prag, 1620. 4to, size $6 \times 7\frac{1}{4}$ inches. This copy of a rare work is rendered unique by the addition of the MS. of Bürgi's introductory matter, which was never printed. It is based on the law of indices. Bürgi thought in algebra; Napier in geometry.

Lent by the Dantzig Town Library.

33. Dr Gieswald, Hustus Byrg als Mathematiker und dessen Einleitung in seine Logarithmen. Dantzig, 1856.

Lent by the Town Library of Dantzig.

VII. The Great Tables preceding the Discovery of Logarithms

34. Opus Palatinum de Triangulis a Georgio Joachimo Rhetico Cœptum: L. Valentinus Otho, Principis Palatini Friderici IV. . . An. Sal. Hum., 1596. 4to, size 8×14 inches. 2 vols.

Lent by the Royal Observatory.

35. Georgii Joachimi Rhætici Magnus Canon Doctrinæ Triangulorum. . . . Neostadii, 1607.

Lent by the Royal Observatory.

36. Thesaurus Mathematicus sive Canon Sinuum ad Radium [10^{15}] . . . Georgio Joachimo Rhetico et cum viris doctis communicatus a Bartholomus Pitisco. . . . Francofurti, 1613. 6mo, size $9\frac{1}{2} \times 14$ inches. Also: Sinus Primi et Postrami Gradus. . . Francofurti, 1613. Also: Principia Sinuum ad Radium . . Auctore Bartholomæo Pitisco. Francofurti, 1613. Also: Sinus Decimorum, Tricesimorum, etc. . . Bartholomæi Pitisci. Francofurti, 1613.

Lent by the Royal Observatory.

37. Canon Triangulorum Emendatissinus . . . Bartholomæi Pitisci. 1608. 4to, size $6 \times 7\frac{1}{2}$ inches. A Table of sines, tangents, secants.

Lent by the Royal Observatory.

Another edition of the same, with Hoffmann's and Jonas Rosa's imprint, and date 1612.

Lent by the Royal Observatory.

38. Bartholomœi Pitisci. . . . Trigonometriæ sive de Dimensione Triangulorum libri quinque. . . . Editio tertia . . . Francofurti, 1612. 4to, size $6\frac{1}{2} \times 8\frac{1}{4}$ inches.

Lent by the Royal Observatory.

39. Trigonometry: or, The Doctrine of Triangles: first written in Latine by Bartholomew Pitiscus . . . trans. by Ra: Handson. 8vo, size $5\frac{1}{2} \times 7\frac{1}{4}$ inches. Also in same volume, A Canon of Triangles. . . . London, 1630.

Lent by the Royal Observatory.

VIII. The Method of Prosthaphæresis

40. Nicolai Raymari Ursi Dithmarsi, Fundamentum Astronomicum: id est, Nova Doctrina Sinuum et Triangulorum. . . . Argentorati, 1588. 4to, size $6 \times 7\frac{1}{2}$ inches.

Lent by the University of Edinburgh.

41. Astronomica Danica Vigiliis & Opera Christiani S. Longomontani . . . elaborata Amsterodami, 1622. 4to, size $7 \times 9\frac{1}{2}$ inches.

Lent by the Royal Observatory.

42. Tychonis Brahe Dani, Epistolarum Astronomicarum Libri. . . . Francofurti, 1610. 4to, size $6\frac{1}{2} \times 8\frac{1}{2}$ inches.

Lent by the University of Edinburgh.

IX. Specimens illustrating Subsequent Developments of Logarithmic Tables

43. Tabulæ Logarithmicæ, or Two Tables of Logarithms. . . . By Nathaniel Roe . . . (and) Edm. Wingate. London, MDCXXXIII. 8vo, size $6\frac{3}{4} \times 4\frac{1}{2}$ inches.

Lent by University College, London.

43*a*. Arithmetiqve Logarithmetiqve, or La Construction & Vsage des Tables Logarithmetiques . . . par Edmond VVingate, gentilhomme Anglois. Paris, MDCXXV. Size $4\frac{1}{4} \times 2\frac{1}{4}$ inches.

Lent by John Spencer, Esq.

44. [Edmund Wingate.] A Logarithmeticall Table. . . . London, 1635. 12mo, size $2\frac{3}{4} \times 3\frac{3}{4}$ inches. In same volume: Artificiall Sines and Tangents. . . .

Lent by the Royal Observatory.

44*a*. New Logarithmes. By John Speidall, and are to be sold at his dwelling house in the Fields. The 7 Impression, 1625.

44*b*. Directorivm Generale Vranometricvm in quo Trigonometriæ Logarithmicæ Fundamenta, ac Regulæ demonstrantur, etc. Authore Fr. Bonaventura Cavalerio, etc. Bononiæ, 1632.

45. Trigonometria Britannica; or, The Doctrine of Triangles, in two books, the one composed and the other translated from the latine copy written by Henry Gellibrand. A Table of Logarithms annexed, by John Newton, M.A. London, 1658. Folio, size $11\frac{1}{4} \times 7\frac{1}{2}$ inches.

Contains translation of Gellibrand's Trigonometria Britannica. 4to, same size.

Lent by the Royal Observatory.

45*a*. Organum Mathematicum libris IX. . . . P. Gaspare. . . . Herbipoli, 1668. 4to, size $6\frac{1}{2} \times 8$ inches.

Lent by J. Ritchie Findlay, Esq.

46. [Henry Sheruris.] Mathematical Tables. . . . London, 1726. (First edition 1705.)

Lent by the Royal Observatory.

46*a*. Sherwin's Tables (Third Edition, 1741).

Lent by Mrs Mary A. Stuart, Duns.

46*b*. A Mathematical Compendium . . . by Sir Jonas Moore, Knight. 4th edition. London, 1705.

Lent by J. Ritchie Findlay, Esq.

47. Geometry Improved . . . by A(braham) S(harp). London, 1717. Folio, size 7×8 inches.

Lent by the Royal Observatory.

48. The Anti-Logarithmic Canon . . . by James Dodson. London, 1742. Folio, size $8 \times 12\frac{1}{4}$ inches.

Lent by the Royal Observatory.

49. Tables of Logarithms . . . by Wm. Gardiner. London, 1742. Folio, size $8\frac{3}{4} \times 10\frac{1}{4}$ inches.

Lent by the Royal Observatory.

50. Tables de Logarithmes . . . par M. Gardiner. Avignon, 1770. Folio, size 12×9 inches.

Lent by the Royal Observatory.

51. Tavole Logarithmiche del Signor Gardiner, corrette da molti Errori. . . . Firenze, 1782. 4to, size $8\frac{1}{4} \times 5\frac{1}{2}$ inches.

Lent by the Royal Observatory.

52. Tables Portatives de Logarithmes . . . par François Callet. Paris, 1795. Stereotyped edition. First edition 1783. 8vo, size $8\frac{1}{2} \times 5\frac{1}{2}$ inches.

Lent by the Royal Observatory.

53. Table of Logarithms, of Sines and Tangents, etc. . . . by F. Callet. Paris, 1795. (Tirage, 1827.)

Lent by the Royal Observatory.

54. Thesaurus Logarithmorum Completus, ex Arithmetica Logarithmica . . Adriani Vlacci collectus. . . A. Georgio Vega. . . . Lipsiæ, in Libraria Weidmannia, 1794. 6mo, size 13×8 inches.

Lent by the Royal Observatory.

55. Tables of Logarithms of all Numbers from 1 to 101000 . . . by Michael Taylor. . . . London, 1792. Folio, size 13×11 inches. Preface by Neville Maskelyne.

Lent by the Royal Observatory.

56. Johann Carl Schulze, . . . Neue und erweiterte Sammlung, logarithmischer, trigonometrischer und anderer . . . Tafeln. 2 Bde. Berlin, 1778. 4to, size $8\frac{1}{2} \times 5$ inches.

Lent by the Royal Observatory.

57. Neue trigonometrische Tafeln . . von Johann Philipp Hobert. Berlin, 1799. 4to, size $8\frac{1}{4} \times 5$ inches.

Lent by the Royal Observatory.

58. Tables Trigonométriques Décimales . . . par Ch. Borda, augmentées et publiées par J. B. J. Delambre. Paris, 1801. 4to, size $9\frac{1}{2} \times 7$ inches.

Lent by the Royal Observatory.

59. Nouvelles Tables Astronomiques et Hydrographiques . . . par V. Bagay. Paris, 1829. 4to, size 10×8 inches. Log. Tables of Trigonometrical Functions are given to single seconds.

Lent by the Royal Observatory.

60. Logarithmic Tables to Seven Places of Decimals . . by Robert Shortrede, F.R.A.S., etc. Edinburgh, 1844. 4to, size $10 \times 6\frac{1}{2}$ inches.

61. Seven-Figure Logarithms of Numbers from 1 to 108000. Proportional Parts by Dr Ludwig Schrön. Fifth edition, corrected and stereotyped, with a description of the Tables added by A. de Morgan. London and Edinburgh: . . . Brunswick, 1865. 8vo, size 10×7 inches. The best of the seven-figure tables in respect to type, paper, arrangement, and general care.

Lent by the Royal Observatory.

62. Tables of Logarithms, by Charles Babbage. London, 1831. Twenty-one volumes of experiments with various coloured papers and inks with the view of finding the least trying combinations.

Lent by the Royal Observatory.

63. A New Table of Seven-Place Logarithms . . . by Edward Sang, F.R.S.E. London, 1871. 8vo, size 7×10 inches.

Lent by the Royal Observatory.

64. Specimen Pages of a Table of the Logarithms of All Numbers up to one million, in preparation by Edward Sang, F.R.S.E. [1872.]

Lent by the Royal Observatory.

65. Nouvelles Tables Trigonométriques Fondamentales . . . par H. Andoyer. Paris, 1911. 4to, size $8\frac{1}{2} \times 11\frac{1}{2}$ inches.

Lent by the Royal Observatory; and by M. Andoyer himself.

66. Tracts on Mathematical and Philosophical Subjects . . . by Charles Hutton, LL.D. and F.R.S., etc. London, 1812. Tract XX. (Vol. i., pp. 306–340), History of Logarithms; Tract XXI. (Vol. i., pp. 340–454), The Construction of Logarithms.

Lent by the Royal Observatory.

II. Dr Edward Sang's Logarithmic, Trigonometrical, and Astronomical Tables. By Cargill G. Knott, D.Sc.

(Reprinted from the *Proceedings* by permission of the Royal Society of Edinburgh.)

At the Council Meeting of 5th July 1907, the following communication was received from the Misses Sang, daughters of the late Dr Edward Sang :—

"We, the daughters of the late Dr Edward Sang, LL.D., F.R.S.E., owners under his will of his collection of MS. Calculations in Trigonometry and Astronomy, having by letter of gift of date 12th February 1906 given the above collection to the President and Council of the Royal Society of Edinburgh, and having, by the cancelling on the 24th May 1907 of their acceptance thereof, received back the collection from the President and Council of the Royal Society of Edinburgh, do hereby give the said collection to the British nation, and do hereby appoint the President and Council of the Royal Society of Edinburgh custodiers of the said collection, in trust for the British nation, with power to publish such parts as may be judged useful to the scientific world.

"We do also hereby give into the custody of the President and Council of the Royal Society of Edinburgh, in trust for the British nation, the duplicate Electrotype Plates of Dr Sang's 1871 New Seven-Place Table of Logarithms to 200,000, with power to use them for reproducing new editions, or publishing extended tables of seven-place logarithms.

"We would express the hope that Dr Sang's idea and plan for reproducing an authoritative and accurate Logarithmic Table, as explained in the last paragraph (p. 6 of the preface to the 1871 New Table of Seven-Place Logarithms), will be borne in mind, and given effect to.

"(Signed) Anna Wilkie Sang.
"(,,) Flora Chalmers Sang.

"Oakdale, Broadstone Park,
Inverness, 1*st July* 1907."

The manuscript volumes number forty-seven in all, the contents of thirty-three of which are in transfer duplicate. Volumes 1 to 3 contain the details of the steps of the calculations on which the results contained in the next thirty-six volumes are based.

Volume 4 contains the logarithms, calculated to 28 figures, of the prime numbers up to 10,000, and a few beyond.

Volumes 5 and 6 contain the logarithms to 28 figures of all numbers up to 20,000.

From these the succeeding thirty-two volumes are constructed, giving the logarithms to 15 places of all numbers from 100,000 to 370,000.

This colossal work must ever remain of the greatest value to computers of logarithmic tables. It is a great national possession.

The other tables in the collection are trigonometrical and astronomical.

Of special interest are the Tables of Sines and Tangents calculated according to the centesimal division of the quadrant.

It is hoped that ere long some of these tables may be published in some form, so as to make them more immediately accessible to computers. They are the foundation of Dr Sang's published book of seven-place logarithms to 200,000, undoubtedly the most perfect of its kind ever printed. By placing the duplicate electrotype plates of this book along with the manuscript volumes in the custody of the Royal Society, with power to publish, the Misses Sang have given to the nation every facility for publishing a new or even an extended edition of their father's work.

The complete account of the various tables follows, and the attention of the scientific world is now drawn to the importance of the collection in the custody of the Society.

In the name of the British nation, the Royal Society of Edinburgh now publicly thank the Misses Sang for their valuable gift, and, as custodiers of these manuscript volumes, undertake to do all in their power to make them of real use to the scientific world.

The above statement was read by the Chairman at the First Ordinary Meeting of the Society, held on 4th November 1907.

The following general account was drawn up in November 1890 by Dr Edward Sang himself :—

"These computations were designed and undertaken with the view to the change from the ancient subdivision of the quadrant to the decimal system, a change long desired, and destined inevitably to be made. One hundred years ago it was on the very point of being completed. Mathematicians were then engaged in the introduction of the decimal system into every branch of calculation and measurement; but for the introduction of this new system into the measurement of angles, it was necessary to have a new trigonometrical canon. The French Government deputed M. Prony, with a large army of computers, to compile this new canon, and astronomers awaited with impatience the advent of this indispensable preparative. Laplace had, in anticipation, reduced all his data in the *Mécanique Céleste* to the new system, and instruments had been graduated suitably.

"We can hardly doubt but that if this new canon had then been published, the decimal graduation of the quadrant would have been very generally adopted even at the beginning of the present century; by the end of the first decade of this century it might indeed have been universally adopted. But the new trigonometrical tables, though magniloquently described, never made their appearance; and thus for something like seventy years the progress of the sciences thereon depending has been impeded.

"Very few are old enough to remember the disappointment felt throughout the scientific world. About 1815, in our school, the boys were exercised in computing short tables of logarithms and of sines and tangents, in order to gain the right to use Hutton's seven-place tables; and well do I recollect the almost awe with which we listened to descriptions of the extent and value of the renowned Cadastre Tables.

"In 1819 the British Government, at the instigation of Gilbert Davies,

M.P., approached the French Government with a proposal to share the expense of publishing the Cadastre Tables, and a commission was appointed to consider the matter. The negotiations, however, fell through, for reasons which were never very publicly made known; but in the session 1820–21 the rumour was current amongst us students of mathematics in the University of Edinburgh, that the English Commissioners were dissatisfied of the soundness of the calculations—and so it was that the idea of an entire recalculation came into my mind.

"In the year 1848, encouraged by the acquisition of a copy of that admirable work, *Burckhardt's Table des Diviseurs* up to three million, the idea took a concrete shape in my mind, and I resolved to systematise the work which before I had carried on in a desultory way. Necessarily the first step was to construct a table of logarithms sufficiently extensive to satisfy all the wants of computers in trigonometry and astronomy; and having many times felt the inconvenience of the loss of the details of the calculations made on separate papers, I resolved to record from the very beginning every important step. This plan of operation has many conveniences—it enables us to retrace and examine every case of doubt, and also to take advantage, in new calculations, of anything in the previous work which may happen to be applicable.

"For all the ordinary operations of surveying and practical astronomy five-place logarithms, as M. Lalande has stated, are perfectly sufficient; and for the higher branches of astronomy and geodetics the usual seven-place tables are enough. But for the purpose of constructing new working tables it becomes necessary to carry the actual work further, both in the extent of the arguments and in the number of decimal places, and therefore I determined on the formation of a table of logarithms to nine places for all numbers up to one million. But again, in order that such a table be true to the ninth place, the actual calculation must be carried still further—and to meet the cases in which the doubtful figures from, say, 4997 to 5003 might occur in one million of cases, it became prudent to carry the accuracy even to the fifteenth place. And this limit of accuracy was further defined by the circumstance that there the differences of the third order just disappear. Even then it may happen that the doubt as to the figures which are to be rejected may not be cleared up, and it follows that a still more minute criterion should be at hand for use, and therefore the order of the work came to be as follows.

"In the first place, the computations of the logarithms of all numbers up to ten thousand, to twenty-eight (for twenty-five) places, was undertaken. At the outset, each logarithm of a prime number was computed twice, but as the work proceeded, it was judged advisable to have three distinct computations of each. The whole of this work is distinctly recorded and indexed, so that every step in reference to any given number can at once be traced out.

"The idea was entertained of this work being ultimately extended to one hundred thousand, and the logarithms of the composite numbers from ten to twenty thousand were computed, spaces being left for those of intermediate prime numbers.

"By the addition of the logarithms thus obtained, those of the great majority of composite numbers from the limit one hundred thousand to one hundred and fifty thousand were computed, and the intervals were filled up by help of second differences. In this part of the work I was aided by my daughters. But, in all such separate additions, we are liable to sporadic errors, and in order to guard against these the whole of this work was redone by the use of the last two figures of the second differences; and thereafter the calculations were made by short interpolations of second differences all the way to three hundred and seventy thousand. Necessarily, on account of the occurrence of the minute final errors, the last, or fifteenth, figures cannot be trusted to within one or two units; and after a very severe examination of the whole, it was found that in a very few instances this accumulation of last-place inaccuracy extended even to five units; and thus we are warranted in expecting that no last-place error will be found reaching so far as to a unit in the fourteenth place—a degree of accuracy far, very far, beyond what can ever be required in any practical matter.

"In the compilation of the trigonometrical canon the same precautions were taken for securing the accuracy of the results. In the usual way, by means of the extraction of the square root, the quadrant was divided into ten equal parts, and the sines of these computed to thirty-three, for thirty places. These again were bisected thrice, thus giving the sine of each eightieth part of the quadrant; all the steps of the process being recorded.

"The quinquesection of these parts was effected by help of the method of the solution of equations of all orders, published by me in 1829; and the computation of the multiples of those parts was effected by the use of the usual formula for second differences. A table of the multiples of 2 ver. $00^c\ 25'$ was made to facilitate the work, and the sines, first differences, and second differences were recorded in such a way as to enable one instantly to examine the accuracy. The same method of quinquesection was again repeated, and the computation of the canon to each fifth minute was effected by help of a table of one thousand multiples of 2 ver. $00^c\ 05'$, the record being given to thirty-three places, the verification being examined at every fifth place. In this work there is no likelihood of a single error having escaped notice.

"For the third time this method of quinquesection was applied in order to obtain the sines of arcs to a single minute. A table of one thousand multiples of 2 ver. $00^c\ 01'$ was computed to thirty-three places, but in the actual canon it was judged proper to curtail these, and the calculations were restricted to eighteen decimals on the scroll paper. In the actual canon as transcribed, only fifteen places are given. In all cases the function, its first difference, and its second difference are given in position ready for instantaneous examination; and the whole is expected to be free of error excepting in the rare cases where the rejected figures are 500—these cases being duly noted.

"For the computation of the canon of logarithmic sines the obvious process is to compute each one of its terms from the actual sine, by help of the table of logarithms; but this process does not possess the great advantage of self-verification, and attempts have been made to obtain a better one.

Formulæ indeed have been given for the computation of the logarithmic sine without the intervention of the sine itself, but when we come to apply these formulæ to actual business we find that they imply a much greater amount of labour than the natural process does; and, after all, they are only applicable to the separate individual cases.

"Nepair, as is well known, arranged his computations of the logarithms from the actual sines in such a way as to lessen by one-half the amount of the labour. Nepair's arrangement was therefore followed, and the work was begun from the sine of 100^c down to 50^c. The calculations were made by help of the fifteen-place table of logarithms from 100,000 to 370,000. If this table had been continued up to the whole million, the labour would have been greatly diminished, but we had to bring the numbers to within the actual range of our table by halving or doubling as the case might be. The results were then tested by first, second, and third differences, and in not a few cases the computation had to be redone, for the sake of some minute difference among the last figures. The log sines for the other half of the quadrant, that is from 50^c to 0^c, were deduced from the preceding by the use of first differences alone. The log tangents from 50^c down to 0^c were also deduced directly by help of the first differences alone. In this way the series of fundamental tables needed for the new system has been completed, so far as the limit of minutes goes.

"While that work was in progress, a circumstance occurred which temporarily changed the order of procedure. Kepler's celebrated problem has ever since his time exercised mathematicians, and, sharing the ambition of many others, I also sought often, and in vain, for an easy solution of it. Accident brought it again before me, and this time, considering not the relations of the lines connected with it, but the relations of the areas concerned, an exceedingly simple solution was found. In order to give effect to this method it was necessary to compute a table of the areas of circular segments in terms of the whole area of the circle. That again rendered it necessary to calculate the sines measured in parts of the quadrant as a unit, instead of in parts of the radius, as usual. This computation was effected by using the multiples of twice the versed sine formerly employed. From this again the canon of circular segments for each minute of the whole circumference was readily deduced. The mean anomaly of a planet may be deduced from its angle of position, or as it is generally called, its excentric anomaly, by simple additions and subtractions of these circular segments. The converse problem is very easily resolved, particularly when the first estimate is a tolerably close one. In order to be able promptly to make this first estimate sufficiently near in every possible case, a table of mean anomalies from degree to degree of the angular position, and also from degree to degree of the angle of excentricity of the orbit, has been computed according to the decimal system.

"The change to this system is inevitable. Each new discovery, each improvement in the art of observing, intensifies the need for the change, at the same time that each augmentation of our stock of data arranged in the ancient way adds to the difficulties. How much the change is needed may be estimated by an inspection of the *Nautical Almanac*. Every page in it

cries out aloud in distress, 'Give us decimals.' For the sun's meridian passage, the usual difference columns are suppressed, and those titled 'var. in 1 hour' are substituted; and similarly for the moon's hourly place a column titled 'var. in 10^m' is given; while for the interpolation of lunar distances, proportional logarithms of the difference are given. While artisans and physicists are using the ten-millionth part of the earth's quadrant as their unit of linear measure, astronomers are still subdividing the quadrant into 90, 60, 60, and 100 parts. The labour of interpolation is unnecessarily doubled at the very least, and that heavy burden is laid on the shoulders of all the daily users of the ephemeris. The trouble attending the reduction of observations tends to lead the navigator to shun the making of observations. The matter is not merely of national, it is of cosmopolitan interest—and this continuous waste of labour has much need to be ended.

"The collection of computations above described contains all that is essentially needed for the change of system, as far as the trigonometrical department is concerned; the great desideratum being the Canon of Logarithmic Sines and Tangents. In addition to the results being accurate to a degree far beyond what can ever be needed in practical matters, it contains what no work of the kind has contained before, a complete and clear record of all the steps by which those results were reached. Thus we are enabled at once to verify, or, if necessary, to correct the record, so making it a standard for all time.

"For these reasons it is proposed that the entire collection be acquired by, and preserved in, some official library, so as to be accessible to all interested in such matters; so that future computers may be enabled to extend the work without the need of recomputing what has been already done; and also so that those extracts which are judged to be expedient may be published.

"Seeing that the Logarithmic Canon is useful in all manner of calculations, the printing of the table of nine-place logarithms might be advantageously proceeded with at once. The publication of the corresponding Canon of Logarithmic Sines and Tangents would only be advisable in the expectation of its early adoption by astronomers.

"But land-surveyors, when transporting the theodolite from one station to another, have to compute the new azimuth from the previously observed one. This is easily done by adding or subtracting 180°; yet in the hurry of business this occasionally gives rise to mistakes. On the other hand, with 400^c on the azimuth circle, we should only have to add or subtract 200^c, thus almost obviating the chance of a mistake. Hence the surveyor would be greatly benefited by the immediate publication of a five-place trigonometrical canon, arranged in the decimal way."

The following 47 volumes of Manuscript Tables are to be seen in the Royal Society Rooms; 32 of these are in transfer duplicate and are on view in the Exhibition Room.

List of Logarithmic, Trigonometrical, and Astronomical Calculations, in Manuscript, by Edward Sang

Nos. 1 *and* 2. *Logarithms I., II. Construction*

The two volumes contain a complete record of the articulate steps of the calculations for the logarithms, to 28 places, of all prime numbers up to 10,000, with those of other large primes which happen in the course of the work.

No. 3. *Logarithms III. Revision*

This third volume contains the calculation, in revision, for all those primes whose logarithms had not been computed thrice. This record is accompanied by an index of all the divisors used in the work, and of the primes themselves and the divisors with which they have been connected. In this revision no deviation exceeding 10 units in the 28th place was allowed to pass.

By this registration, a future computer is enabled to lessen his labour when he happens to have to do with a divisor which had occurred before, or when any easy multiple or submultiple may occur.

No. 4. *Logarithms. Primes*

This is a list of the first 10,000 prime numbers (up to 104,759), with the logarithms, to 28 places, of those which have been computed (continuously up to 10,037, with occasional ones beyond), and with references to the pages of the construction in which they have been given. (The logarithms of the remaining primes are given to 15 places.)

No. 5. *Logarithms* 0

Contains the logarithms, to 28 places, of all numbers up to 10,000; those of the composites having been got by the addition of those contained in No. 4.

No. 6. *Logarithms I.*

Contains the logarithms, to 28 places, of all composite numbers from 10,000 to 20,000, with those of primes incidentally found.

Nos. 7, 8, 9, 10, 11. *Logarithms* 10, 11, 12, 13, 14
(*Nos.* 100,000 *to* 150,000)

The logarithms given in these five volumes are restricted to 15 places. Those of the majority of the composite numbers were got by addition from vols. 0 and 1; the intermediates having been filled in by interpolation of second differences. This work had been done on scroll paper, and thence copied on the actual pages.

Nos. 12, 13, 14, 15, 16. *Logarithms* 10, 11, 12, 13, 14
(*Nos.* 100,000 *to* 150,000)

In order to remove the risk of detached errors in copying, the last two figures of the second differences were alone copied into their places from the

previous volumes, and from these the complete second differences, the first differences, and the logarithms were re-computed by integration. (Also in transfer duplicate.) [1]

Nos. 17, 18, 19, 20, 21. *Logarithms* 15, 16, 17, 18, 19
(*Nos.* 150,000 *to* 200,000)

The logarithms in these five volumes were got by interpolating two terms between the even numbers of the preceding volumes, adding the logarithm of 1·5. The interpolation was done on paper-aside, using only the last two figures of the second differences. These last two figures were then copied into their places on the actual pages, and the work finished by integration. (Also in transfer duplicate.)

Nos. 22–38. *Logarithms* 20–36 (*Nos.* 200,000 *to* 370,000)

In these seventeen volumes, the logarithms have been found by interpolating one term between the terms of the preceding volumes from 10, adding the logarithm of 2; the work having been done by integration as before, and the results tested by addition at least twice in each decade. (Also in transfer duplicate.)

No. 39. *Logarithms. Auxiliary Table*

This volume shows the last 10 figures of the logarithms of numbers from 1 00000 0000 to 1 00000 9999, and from 1 00000 0000 to 99999 0000, which are used for computing the logarithms of numbers consisting of more than six effective places. (Also in transfer duplicate.)

No. 40. *Sines*

This is the record of all the articulate steps in the calculation, to 33 places, of the sines of arcs differing by the 2000th part of the quadrant.

By the extraction of the square root and repeated bisections, the quadrant was divided into eighty parts, and the sines of the multiples of $\cdot 01^c\ 25'$ were computed.

Thereafter the sines and cosines of $00^c\ 25'$ and of $01^c\ 25'$ were got by the direct resolution of the appropriate equations of the fifth degree, and were compared with those which had been got in computing the recurring functions of submultiples of π, the steps of which are copied into this record.

By help of 100 multiples of 2 ver. 25′, and of 1000 multiples of 2 ver. 5′, a table of sines of arcs differing by 25′, and thereafter one of arcs differing by 5′, were computed on the actual pages.

Although these have the appearance of being interpolations, they are truly independent computations, the use of the preceding work preventing mistakes, as well as the accumulation of the minute errors due to the rejection of figures beyond the 33rd place.

[1] The volumes in transfer duplicate have been placed in the library of the University of Edinburgh.

Nos. 41, 42. *Canon of Sines, Parts I., II.*

These volumes contain the sines to 15 places of arcs differing by 1′ (centesimal division) with their first and second differences, the computation having been facilitated by a table of 1000 multiples of 2 ver. 1′.

The table has been bound in two parts, for the convenience of referring to the sine and to the cosine of an arc. (Also in transfer duplicate.)

No. 43. *Log Sines and Tangents*

The log sines from $100^c\ 00'$ down to $50^c\ 00'$ are here given to 15 places, with their first, second, and third differences. They were computed directly from the Canon of Sines by the 15-place table of logarithms from 100 000 to 370 000, and by use of the auxiliary table.

The log sines from $50^c\ 00'$ to $0^c\ 00'$ were derived from the preceding, according to the formula—

$$\sin a = \tfrac{1}{2} \sin 2\,a.\ \sec a,$$

using the first differences only.

The log tangents from $50^c\ 00'$ to $0^c\ 00'$ were obtained from the preceding log sines, using only the first differences.

Upwards of two million eight hundred thousand figures were written for the completion of this volume. (Also in transfer duplicate.)

No. 44. *Sines in Degrees*

This volume contains the values of the sines measured, not in parts of the radius, but in parts of the quadrant, and given to the ten-thousandth part of the degree. These sines were computed directly from degree to degree, then for each quarter of a degree, using the multiples of 2 ver. 25′, then to each 20th of a degree, and lastly to each minute. The work thus represents three independent computations.

No. 45. *Circular Segments*

These circular segments are measured in parts of the surface of the circle as divided into 400 degrees of surface, and these subdivided into 1 0000 0000 parts. They have been computed by the integration of the second differences of the sines measured in degrees, and are carried round the entire 400 degrees of the circumference.

This table is intended to facilitate calculations concerning the elliptic motions of the planets; it gives us the mean anomaly when the planet's position is given, from the formula—

$$\text{Mean anomaly} = \tfrac{1}{2}\{\text{segm}\ (p+e) + \text{segm}\ (p-e)\},$$

in which p is the angle of position and e the angle of eccentricity of the orbit. (Also in transfer duplicate.)

No. 46. *Mean Anomalies* (*A*)

These are the mean anomalies in orbits of each degree of eccentricity from $e = 0^c$ to $e = 100^c$, given for each arc of position from $p = 0^c$ to $p = 200^c$, and carried to the eighth decimal place of the degree.

No. 47. Mean Anomalies (B)

In this volume the anomalies are given only to the nearest second, but the differences for a change of 1^c of position, and the variations for a change of 1^c in ellipticity, are filled in; and thus, of the three—the eccentricity, the position, the anomaly—any one may be determined from the others. (Also in transfer duplicate.)

III. A Working List of Mathematical Tables. By Herbert Bell, M.A., and J. R. Milne, D.Sc.

The object of the following list of mathematical tables is a purely practical one. It is to afford the computer a ready means of ascertaining what functions have been tabulated, and the ranges over which the tabulation extends. In order the better to do this, such considerations as the historic interest of the tables, their chronological order of publication, and the like, have been for the most part ignored. Again, in many cases the mention of tables has been omitted on the ground that they are now unlikely to be useful to those actually engaged in calculation, having for one reason or another been superseded by others of later date. Also, no notice has been taken of the more popular tables, such as those published for school use; for instance, under the heading "Logarithm Tables" none is mentioned having a less accuracy than seven places. Tables of mathematical functions which are of purely technical application have been omitted, because they are only of interest to a limited number of persons, who are probably well aware of their existence. Hence such tables as those relating to surveying, spherical co-ordinates, engineering formulæ, etc., have been left out, and in order to keep the list as compact as possible and facilitate readiness of reference, no more information has been given in each case than serves to identify the particular table mentioned.

The authors beg to acknowledge their indebtedness to such sources of information as the *International Catalogue of Scientific Literature*, the *British Association Reports*, the *Jahrbuch über die Fortschritte der Mathematik*, the article by R. Mehmke in *Numerisches Rechen*, vol. i. pt. ii. pp. 941–1079 of *Encyk. der math. Wiss.* (Leipzig, 1900–4).

Especial mention should be made, however, of the exhaustive article on Mathematical Tables by Dr Glaisher in the *Encyclopædia Britannica*. For some of the more special tables little has been done beyond embodying his information.

FACTOR TABLES

Nine quarto volumes (*Factor Tables*, London, 1829–1883) form a uniform table giving least divisors of all numbers in the first nine millions not divisible by 2, 3, or 5.

141 errata by J. P. Gram (*Acta Math.*, 1893).

Lehmer's table (Carnegie Institution, 1909) gives least factor of all numbers up to 10 millions not divisible by 2, 3, 5, or 7.

Vienna Academy has MSS. of factor tables to 100 millions (see *Ency. math. Wissens.*, 1900–4, 952).

Tables relating to the Theory of Numbers :—

These are too highly technical to be mentioned here in detail. Reference should be made in the first instance to Glaisher's article "Tables" in the *Encyclopædia Britannica*, where references to standard literature on the subject will be found.

ARITHMETICAL TABLES

Multiplication :—

(1) Direct :—

To 999×999—

Crelle's Tables. Frequent editions in English, French, and German.

To 99×9999—

Zimmermann, *Rechen Tafeln* (Berlin); Peters (Berlin, 1909).

(2) By quarter squares, using formula $ab = \frac{1}{4}(a+b)^2 - \frac{1}{4}(a-b)^2$. Up to $99{,}999 \times 99{,}999$ by using *Table of Quarter Squares up to* 200,000, by J. Blater (Trübner, London).

Squares, Cubes, Roots, etc. :—

Squares and Cubes to 100,000, by J. P. Kulik (Leipzig, 1848); also by Blater (see above) to 200,000. Squares to 100,000 by Laundy (London, 1856).

Square and Cube Roots up to 25,500, *below* 1010 *to* 14 *decimals, above to* 5, G. E. Gélin (Huy, 1894).

Barlow's *Tables* in the usual editions give squares, cubes, square and cube roots, and reciprocals to 10,000. The first edition (1814) also contained higher powers.

Reciprocals :—

Oakes (Layton, London) and *Cotsworth's Direct Reciprocals* (M'Corquodale & Co., Leeds) give to seven significant figures the reciprocals of all numbers to 10 millions.

Factorials :—

$\text{Log}_{10}\, n\,!$ from $n=1$ to $n=1200$ to eighteen places are given by C. F. Degen, *Tabularum Enneas* (Copenhagen, 1824). Shortrede, *Tables* (1849, vol. i.) gives $\log n\,!$ to five places up to $n=1000$.

$n \times n\,!$ to twenty figures and
$-\log\,(n \times n\,!)$ to ten places

are given by Glaisher as far as $n=71$ in *Phil. Trans.* (1870, p. 370), and $1/n!$ to twenty-eight figures as far as $n=50$ in *Camb. Phil. Trans.*, xiii. p. 246.

TABLES FOR THE SOLUTION OF EQUATIONS

Quadratic Equations :—

R. Mehme in *Schlömilch's Zeitsch.*, 1898, xliii. p. 80.

Cubic Equations :—

The values of $\pm(x - x^3)$ are given by J. P. Kulik, *Abh. d. k. Böhm. Ges. d. Wiss. Prague*, 1860, xi. pp. 1–123. From $x = 0{\cdot}0000$ to $x = 3{\cdot}2800$ to seven places.

S. Gundelfinger, *Taf. zur Berechnung d. reelen Wurzeln sämtlichen trinomischen Gleichungen* (Leipzig, 1897). This also deals with equations of the fifth order.

Transcendental Equations :—

Some roots of the following equations :—

$$\tan x = x$$

$$\tan x = \frac{2x}{2 - x^2}$$

$$\cos x \cosh x = \pm 1$$

$$\tanh x = -\tan x$$

are given, together with account of sources, in Jahnke and Emde.[1]

BINOMIAL THEOREM COEFFICIENTS

That is, values of

$$\frac{x(x-1)}{1.2.}, \quad \frac{x(x-1)(x-2)}{1. \quad 2. \quad 3.} \text{ etc.}$$

for various values of x.

The first two are given in Dale[2] and in Chambers, from ·01 to 1·00 to three places.

The first fifteen are given by Lambert, *Supplementa* (1798), for $x = \frac{1}{2}$.

The first five to seven places from ·01 to 1·00 are given by Barlow (1814) and Köhler (1848).

All values, for integral values of x as far as forty, are given with their logs to seven places by H. Gyldén (*Receuil des Tables*, Stockholm, 1880).

TABLES OF CONSTANTS

Euler's constant has been calculated to 263 places by J. C. Adams, *Proc. Roy. Soc.*, xxvii., p. 88.

Functions of π :—

π^n where n has various values, integral and fractional, is given in most collections of tables. A large number (71) of such constants is given in W. Templeton's *Millwright's and Engineer's Pocket Companion* (London) to about thirty places. See also G. Paucker, *Grunert's Archiv*, vol. i. pp. 9–10 ; Glaisher, *Proc. Lond. Math. Soc.*, viii. p. 140, and J. P. Kulik, *Tafel d. Quad. u. Kubik-Zahlen* (Leipzig, 1848). π itself has been calculated by Shanks (*Proc. Roy. Soc.*, xxi. p. 319) to 707 places.

$e^{n\pi}$, see under e^x.

The series :

$$S_n = 1^{-n} + 2^{-n} + 3^{-n} + \text{ etc.}$$
$$s_n = 1^{-n} - 2^{-n} + 3^{-n} - \text{ etc.}$$
$$\sigma_n = 1^{-n} + 3^{-n} + 5^{-n} + \text{ etc.}$$
$$\Sigma_n = 2^{-n} + 3^{-n} + 5^{-n} + \text{ etc. (primes only)}$$

[1] Jahnke und Emde, *Funktiontafeln* (Teubner, Leipzig, 1909). A very thorough book, well illustrated graphically. It should be consulted for tables of most of the higher functions.

[2] Dale's *Five-Figure Tables* (Edward Arnold, London, 1903). A small and convenient collection which contains a considerable number of tables of transcendental functions.

are tabulated (and in large measure calculated) by Glaisher for various integral values of n in *Proc. Lond. Math. Soc.*, viii. p. 140, and in *Compte rendu de l'Ass. Française*, 1878, p. 172. A small but convenient table is given in Dale. For further information consult Glaisher's article in *Encyc. Brit.*

Bernoullian Numbers :—

The first sixty-two are published by J. C. Adams in *British Association Report* for 1877 and in *Crelle's Journal*, lxxxv. p. 269. The first nine figures of the first 250 numbers and their logarithms are given by Glaisher, *Cambridge Phil. Trans.*, xii. p. 384.

LOGARITHMS TO BASE "e"

These, although commonly called "Napierian logarithms," were first published by J. Speidell in *New Logarithmes* (1619). Napier's logarithms were to the base $1/e$.

To seven places :—

Barlow (London). From 1 to 10,000.

Z. Dase (Vienna, 1850). From 1 to 10,000, and, at intervals of ·1, from 1000 to 10,500. This is the most extensive table.

Dupuis (Paris, 1912). From 1 to 1000.

Hutton (London). From 1 to 1200.

Willich (1853). From 1 to 1200.

To eight places :—

J. Hantschl, *Log.-trig. Handbuch* (Vienna, 1827). From 1 to 11,273.

Köhler (1848) and

Vega, *Tabulæ* (Leipzig, 1848), which includes Hülsse's 1840 edition, from 1 to 1000 and primes as far as 10,000.

Ree's *Cyclopædia* (1827), art. "Hyperbolic Logarithms," from 1 to 10,000.

To ten places :—

Salomon (Vienna, 1827). From 1 to 1000, and primes as far as 10,333.

To eleven places :—

Borda and Delambre (Paris, 1801). From 1 to 1200.

To forty-eight places :—

Callet (Paris). From 1 to 100, and primes as far as 1097.

Vega's *Thesaurus* (Leipzig, 1794, reprinted Milan, 1909) gives Wolfram's logarithms of numbers from 1 to 2200, and of primes to 10,009.

W. Thiele (Dessau, 1907) recalculated and extended in certain details Wolfram's logarithms.

Adams, in *Proc. Roy. Soc.*, 1886, xlii. p. 22, gives the logs of 2, 3, 5 and 7 to 276 places, and those of log 10 and its reciprocal to 272 places.

TABLES FOR CONVERTING FROM BASE 10 TO BASE e, AND VICE VERSA

Multiples of the conversion factor are given :—

To seven places by BREMIKER (Berlin, 1906), and DUPUIS (Paris, 1912).

To ten places by SCHRÖN (Braunschw., also Eng. ed.), and BRUHNS (Leipzig).

To thirty places by DEGEN, *Tabularum Enneas* (Copenhagen, 1824).

See ADAMS under " Logs to Base *e*."

TABLES OF LOGARITHMS TO BASE 10

The original calculations of the larger canons are contained in :

BRIGGS' *Arithmetica logarithmica* (London, 1624), to fourteen places from 1 to 20,000 and from 90,000 to 100,000, and

VLACQ'S *Arithmetica logarithmica* (Gouda, 1628), to ten places from 1 to 100,000, reissued by Vega in 1794 at Leipzig in the *Thesaurus logarithmorum completus.*

The following is a list of some of the larger accessible tables arranged in order of accuracy :—

To seven places :—

BABBAGE (London, 1889). From 1 to 108,000. Contains only logs of numbers.

C. BREMIKER (Berlin, 1906). From 10,000 to 11,000.

C. BRUHNS (Leipzig, 1906). Collection of tables.

F. CALLET, *Tables portatives* (Paris). From 1 to 108,000. A collection of Tables.

CHAMBERS' *Mathematical Tables* (Edinburgh). From 1 to 108,000.

DIETRICHKEIT (Berlin, 1906). Contains also seven-figure antilogs.

J. DUPUIS (Paris, 1912), *Tables des logarithmes.* From 1 to 100,000. A collection of tables.

LALANDE (Paris, 1907).

J. SALOMON (Vienna, 1827). From 1 to 108,000.

SANG (London). From 20,000 to 200,000.

L. SCHRÖN. From 1 to 108,000. A collection of tables.

SHORTREDE, *Logarithmic Tables* (Edinburgh). From 1 to 120,000. Gives also the means for finding logarithms and antilogarithms to sixteen and twenty-five places.

To eight places :—

BAUSCHINGER AND PETERS (Leipzig, 1909). From 1 to 200,000.

MENDIZÁBAL TAMBORREL (Paris, 1891). From 1 to 125,000.

J. NEWTON, *Trigonometrica Britannica* (London, 1658). From 1 to 100,000.

SERVICE GÉOGRAPHIQUE DE L'ARMÉE (Paris, 1891). From 1 to 120,000.

To ten places :—

BRIGGS' (see above).

W. W. DUFFIELD, *Report of the U.S. Coast and Geodetic Survey* (Washington, 1895–6, App. xii.). From 1 to 100,000.

Erskine Scott (Layton, London). Also gives antilogs.
Vega (Milan, 1909). New edition of the *Thesaurus*.
Vlacq, see above.

To eleven places :—
Börgen (Leipzig, 1907). From 1 to 100.
Delezenm (Lille, 1857).

To twelve places —
Namur (Brussels, 1877).

To twenty places :—
In Hutton's seven-place tables as far as 1200.

To twenty-seven places :—
Thoman (Paris, 1867).
Abraham Sharp, *Geometry Improv'd*, 1717. Gives to sixty-one places logs of all numbers from 1 to 100 and of all primes from 100 to 1100.

TABLES OF ANTILOGARITHMS

To seven figures :—
Dietrichkeit (Berlin, 1906),
Filipowski (London, 1849), and
Shortrede (Edinburgh, 1849) for all five-place decimal fractions.

To eleven figures :—
J. Dodson, *Anti-logarithmic Canon* (London, 1742). The earliest and largest work, for all five-place decimal fractions.

To twenty figures :—
Shorter tables are given by Gardiner (1742), Callet, and Hutton.

GAUSSIAN LOGARITHMS

These give the value of log $(a+b)$ or log $(a-b)$ when log a and log b are given separately. Usually the argument (D) is log $a-$log b. There are several modifications.

The first suggestion for such tables, together with a specimen page, was made by Leonelli, *Théorie des logarithmes* (Bordeaux, 1803), reprinted by J. Houël (Paris, 1875). The first table is by Gauss, *Zach's Mon. Corresp.* (1812), reprinted in *Werke*, vol. iii. p. 224. It is a five-place table.

To six places :—
B. Cohn, *Tafeln* (Leipzig, 1909), a very convenient table, mostly at intervals of ·001 in D.
Bremiker, *Sechstellige Log.* (Leipzig), about the same intervals.
Gray, *Tables and Formulæ* (London, 1870). Tabulates at intervals of ·0001.
Gundelfinger, *Sechstell. Gauss* . . . (Leipzig, 1902), at intervals of ·001.
G. W. Jones, *Logarithmic Tables* (London and Ithaca, N.Y., 1893), at intervals of ·001.

To seven places :—

MATTHIESEN, *Tafel zur bequemern Berechnung* (Altona, 1818), at intervals of ·0001. Not convenient.

T. WITTSTEIN, *Logarithmes de Gauss* (Hanover, 1866), and

J. ZECH, *Tafeln der Add. u. Subtr.-Log.* (Leipzig, 1849). Both tabulate in convenient form at intervals of the order of ·0001.

TABLES FOR USE IN CALCULATING LOGARITHMS, ETC.

BÖHM (Vienna, 1880). See also *Astr. Nachr.*, 1910. Tables to enable one to calculate logarithms to twenty places.

FRISCHAUF. Note on accuracy of Steinhauser's table. *Astron. Nachr.*, 174, Nr. 4163, 173–4.

GRAY (Layton, London, 1876). Tables for calculating logs and anti-logs to twenty-four places.

GUILLEMIN (Paris). Log tables equivalent to logs to six and to nine places.

GUNDELFINGER and NELL (Darmstadt, 1911). Tables for calculating nine-figure logarithms.

HOPPE, *Tafeln* (Leipzig, 1876). Tables for calculating thirty-figure logs.

KRAMER, J. Application of differences to calculation of tables (Bl., Berlin, vi., 1909).

S. PINETO, *Tables de Logarithmes* . . . (St. Petersburg, 1871). Tables enabling one to get ten-figure logs.

STEINHAUSER, *Hilfstafeln* (Vienna, 1880). Tables for calculating logarithms to twenty places.

WOODWARD. Tables to aid in map-making (Washington, 1899).

LOGARITHMIC TRIGONOMETRICAL FUNCTIONS

The two great original tables are :—

VLACQ'S *Trigonometria artificialis* (Gouda, 1633), giving log sines and tangents for every ten seconds to ten places, and

BRIGGS' *Trigonometria Britannica* (London, 1633), giving log sines to fourteen places and log tangents to ten places for every hundredth of a degree to 45°.

H. ANDOYER (Paris, 1911) gives log trig. functions for every tenth sexagesimal second to fourteen places.

V. BAGAY (Paris, 1829) gives log trig. functions to seven places for every sexagesimal second.

BAUSCHINGER and PETERS (Leipzig, 1911) give log trig. functions for every sexagesimal second to eight places.

CHAMBERS' *Mathematical Tables* gives log trig. functions to seven places for every sexagesimal minute.

J. PETERS (Leipzig, 1911) gives log trig. functions for every sexagesimal second to seven places.

SHORTREDE, *Logarithmic Tables* (Edinburgh) (revised edition by Hannyngton Layton, London), gives log sines and tangents to seven places for every sexagesimal second.

C. BREMIKER, *Log.-trig. Tafeln* (Berlin, 1906), gives to five places log trig. functions for every hundredth of a degree (following Briggs' method).

J. P. HOBERT and L. IDELER, *Nouvelles tables trigonométriques* (Berlin, 1799) and

C. BORDA and J. B. J. DELAMBRE, *Tables trigonométriques décimales* (Paris, 1810), give log trig. functions to seven places for every centesimal minute.

SERVICE GÉOGRAPHIQUE DE L'ARMÉE, *Tables des logarithmes à huit décimales* . . . (Paris, 1891), gives log sines and tangents for every ten centesimal seconds to eight places.

SERVICE GÉOGRAPHIQUE DE L'ARMÉE, *Nouvelles tables de logarithmes* (Paris, 1906), gives to five places log trig. functions in both the centesimal and the sexagesimal systems.

BECKER and VAN ORSTRAND [1] give log trig. functions to five places for every ·001 radians in first quadrant.

MENDIZÁBAL TAMBORREL, *Tables des logarithmes* (Paris, 1891), gives log trig. functions to eight places for every 10^{-6} gone (about 1·3 sexagesimal seconds). A *gone*=360°.

NATURAL TRIGONOMETRICAL FUNCTIONS

All later tables are abridgments of the great tables by Rheticus, the *Opus Palatinum* (Neustadt, 1596) giving all the trigonometrical ratios for every ten seconds to ten places, and the *Thesaurus Mathematicus* (Frankfurt, 1613) giving the natural sines for every ten seconds to fifteen places with first, second, and third differences.

These were calculated just before the invention of logarithms, and Rheticus is said to have had computers at work for twelve years.

CHAMBERS. Nat. trig. functions for every sexagesimal minute to seven places.

E. GIFFORD, *Natural Sines* (Manchester, 1914), gives the natural sines for every sexagesimal second to eight places.

J. PETERS (Berlin, 1911). Sines and cosines enabling one to read to twenty-one places for every sexagesimal second.

LOHSE (Leipzig, 1909). Nat. trig. functions to five places for every hundredth of a sexagesimal degree.

HOBERT and IDELER, *Nouvelles tables trigonométriques* (Berlin; 1789). Nat. trig. functions to seven places for every centesimal second.

BECKER and VAN ORSTRAND. Nat. trig. functions to five places for every ·001 of a radian.

BURRAU (Berlin, 1907). Nat. trig. functions to six places for every ·01 of a radian.

[1] Becker and Van Orstrand (*Smithsonian Mathematical Tables*, Washington, 1909). This book should be consulted for any table involving Hyperbolic Functions.

CHANGE FROM ONE SYSTEM OF ANGULAR MEASURE TO ANOTHER

CHAMBERS' *Mathematical Tables.* Degrees and minutes in first quadrant to radians, and on each page the necessary differences for any number of seconds to seven places (Edinburgh, 1893).

DALE gives a convenient table to five places for converting from degrees, minutes, and seconds to radians.

BECKER and VAN ORSTRAND give a short table for converting from degrees, etc., to radians to eleven places, and from any five-place decimal fraction of a radian to seconds to seven places.

HYPERBOLIC AND EXPONENTIAL FUNCTIONS

Log$_{10}$ *sinh x, log*$_{10}$ *cosh x* :—

Gudermann gave tables for the quadrant at intervals of ·01 of a grade to seven places. He also gave a nine-place table from $x=2{\cdot}500$ to $x=5{\cdot}000$, and a ten-place table from $x=5{\cdot}00$ to $x=12{\cdot}00$. Ligowski, *Tafeln der Hyperbelf.* (Berlin, 1890) fills the gap from $x=0{\cdot}000$ to $x=2{\cdot}000$, using five places. He also evaluates from $x=2{\cdot}00$ to $x=9{\cdot}00$. Becker and Van Orstrand give logs to five places from $x=0{\cdot}0000$ to $x=0{\cdot}1000$, from $x=0{\cdot}100$ to $x=3{\cdot}000$, and from $x=3{\cdot}00$ to $x=6{\cdot}00$.

Sinh x and cosh x :—

Ligowski gives these to six places from 0·00 to 8·00. Burrau (Berlin, 1907) gives them from 0·00 to 10·00 to five places; Dale from 0·00 to 2·00 and from 2·0 to 6·0 to five places; Becker and Van Orstrand for same arguments and to same accuracy as their logarithms. See also under e^x.

Log e^x :—

Given by Glaisher, *Camb. Phil. Trans.*, xiii. 1883, from $x=0{\cdot}000$ to $x=0{\cdot}100$, from 0·00 to 2·00, from 0·0 to 10·0, and at unit intervals to 500, all to ten places. Dale gives same from 1·0 or 10·0 to five places. Becker and Van Orstrand give seven-place values from 0·000 to 3·000 and from 3·00 to 6·00.

e^{-x} is given by F. W. Newman, *Camb. Phil. Trans.*, xiii. 1883, from 0·000 to 15·349 to eighteen places; from 15·350 to 17·298 (at intervals of ·002), and thence (at intervals of ·005) to 27·635 to fourteen places. It is given by Becker and Van Orstrand for same range and accuracy as log e^x.

e^x is given by Glaisher, *Camb. Phil. Trans.*, xiii. 1883, to nine figures for same arguments as log e^x. It is also given by Dale, and Becker and Van Orstrand for same arguments and accuracy as e^{-x}. It is given by Van Orstrand, *Tables of the Expon. Functions* (Washington, 1913) from $x=0{\cdot}0$ to $x=32{\cdot}0$, along with the corresponding value of e^{-x}, to twenty places. The corresponding values of sinh x and cosh x can, of course, be easily deduced.

e^n, $e^{\cdot n}$, $e^{\cdot 0n}$, . . . $e^{\cdot 0000000n}$, where n has values 1, 2, 3, . . . 9, are given in Salomon's *Tafeln*, 1827. $e^{n\pi}$ where n has various integral and functional values is given by Gauss (*Werke*, vol. iii.) to about fifty places. A considerable table is also given by Dale.

$\frac{2}{\sqrt{\pi}}e^{-x^2}$ is given by J. Burgess with the same arguments and accuracy as its integral (*q.v.*), *Trans. Roy. Soc. Edin.*, 1888, XXXIX. ii., No. 9.

$\frac{e^x}{\sqrt{\frac{1}{2}\pi x}}$ and $\frac{e^{-x}}{\sqrt{\frac{1}{2}\pi x}}$ are tabulated in Jahnke and Emde to four places from 0·0 to 6·0.

MISCELLANEOUS ELEMENTARY FUNCTIONS

ASTRAND. Kepler's problem, tables for solving (Leipzig, 1890).

CHAMBERS. Areas of segments of a circle of unit diameter and of heights from ·001 to ·500.

DITTMANN (Wurzburg, 1859). *Co-ordinate* tables for expressing x and y in terms of r and θ to seven places.

FARLEY. (London Nautical Almanac Office, 1856.) Natural versed sines from 0° to 125° and log versed sines from 0° to 135°.

HANNYNGTON. (London, 1876). Log haversines from 0° to 180° for every 15 seconds and natural haversines from 0° to 180° for every 10 seconds, all to seven places.

HAUSSNER. Table for Goldbach's Law (Halle, 1897).

PASQUICH, *Tabulæ Logarithmico-trigonometricæ* (Leipzig, 1817), gives $\sin^2 x$, $\cos^2 x$, $\tan^2 x$, $\cot^2 x$ from 1° to 45° at intervals of 1 minute to five places.

SCHLESINGER. Computation of $u - \sin u$.

TABLES CONNECTED WITH ELLIPTIC FUNCTIONS

$$F(k, \phi) = \int_0^\phi \frac{dx}{\sqrt{1 - k^2 \sin^2 x}}; \quad E(k,\phi) = \int_0^\phi \sqrt{1 - k^2 \sin^2 x}\,dx \quad (k = \sin\theta).$$

Denoting the modular angle by θ, the amplitude by ϕ, the incomplete integral of the first and second kind by $F(\phi)$ and $E(\phi)$, and the complete integrals by K and E, we have the following tables by Legendre in vol. ii. of his *Traité des fonctions elliptiques* (1826) :—

(1) $\log_{10}$ E and $\log_{10}$ K from $\theta = 0$ to $\theta = 90°$ at intervals of 0°·1 to twelve or fourteen places with differences to the third order; (2) $E(\phi)$ and $F(\phi)$, when the modular angle is 45°, from $\phi = 0°$ to $\phi = 90°$ at half-degree intervals to twelve places with differences to the fifth order; (3) E (45°) and F (45°) from $\theta = 0°$ to $\theta = 90°$ at intervals of one degree with differences to sixth order, also E and K to same order all to twelve places; (4) $E(\phi)$ and $F(\phi)$ for every degree of both the amplitude and argument to nine or ten places.

Extensive tables for E and F to four places are given in Jahnke and Emde.

q–Tables :—

$\log_{10}(\log_{10} q^{-1})$ argument θ at intervals of 0°·1 to twelve or fourteen places by P. F. Verhulst, *Traité des fonctions elliptiques* (Brussels, 1841).

$\log_{10} q$ from $\theta = 0$ to $\theta = 90°$ as follows: Glaisher in *Month. Not. R.A.S.* (1877), xxxvii. p. 372, for every degree to ten places; C. S. Jacobi in *Crelle's*

Journal, xxvi. p. 93, for every tenth of a degree to five places; J. Bertrand in his *Calcul Intégral* (1870), for every five minutes to five places, and Jahnke and Emde for every five minutes to four places; E. D. F. Meissel, in his *Sammlung mathematischen Tafeln* (Iserlohn, 1860), for every minute to eight places. A useful compendium of elliptic function tables is published by Bohlin (Stockholm, 1900).

Theta functions, etc. :—

Tables to a considerable extent are reproduced in Jahnke and Emde, together with information about existing tables.

LEGENDRE AND BESSEL FUNCTIONS

Legendrian Coefficients or Zonal Harmonics :—

The values of $P_n(x)$ from $x=0$ to $x=1$ at intervals of ·01 and from $P_1(x)$ to $P_7(x)$ are given by Glaisher in *Brit. Assoc. Rep.*, 1879, pp. 54–57. They are reproduced in Dale, and in Jahnke and Emde.

$P_n(\cos\theta)$ for $n=1, 2, \ldots 7$, and for $\theta=0°, 1°, \ldots 90°$ are given to four places by J. Perry in *Proc. Phys. Soc.*, 1892, ii. p. 221, and in *Phil. Mag.*, 1891, series 6, xxxii. p. 512. They are reproduced in Jahnke and Emde.

$\frac{d}{d\theta}P_1(\theta), \frac{d}{d\theta}P_2(\theta) \ldots \frac{d}{d\theta}P_7(\theta)$ are given in Jahnke and Emde to four places for every degree of the quadrant. The most complete tables of Legendrian and associated Legendrian functions were given by Tallqvist at Helsingfors, 1908.

Bessel's Functions :—

P. A. Hansen's extension of Bessel's tables is reproduced by O. Schlömilch in *Zeitsch. für Math.*, ii. p. 158, and by E. Lommel, *Studien über die Bessel'schen Functionen*, Leipzig (1868), p. 127. It gives $J_0(x)$ and $J_1(x)$ from $x=0$ to $x=20$ at intervals of ·01 throughout the lower part of the range, as well as $J_n(x)$ for various values of n up to 28, all to seven places.

$J_0(x)$ and $J_1(x)$ from $x=0$ to $x=15{\cdot}50$ at intervals of ·01 are given by E. D. F. Meissel in the *Abh. d. Berlin. Akademie*, 1888, and in Jahnke and Emde.

$I_n(x)=i^{-n}J_n(ix)$. These tables are given by A. Lodge, *Brit. Assoc. Report*, 1889, p. 29, for $n=0, 1, \ldots 11$ from $x=0$ to $x=6$ at intervals of 0·2 to eleven or twelve places. $I_0(x)$ and $I_1(x)$ are given at intervals of ·001 to nine places from $x=0$ to $x=5{\cdot}100$ (*id.*, 1893, p. 229, and 1896, p. 99).

Subsidiary tables for the calculation of Bessel's functions are given by L. V. G. Filon and A. Lodge in *Brit. Assoc. Rep.*, 1907, p. 94. The work is being continued, the object being to tabulate $J_n(x)$ for $n=0, \frac{1}{2}, 1, 1\frac{1}{2}, \ldots 6\frac{1}{2}$. For the list of all tables before 1909 connected with Bessel's functions and very complete sets of tables see Jahnke and Emde. Six-figure tables of Bessel functions for imaginary arguments are given by Anding (Leipzig, 1911).

$J_0(x\sqrt{i})$ at intervals of 0·2 from $x=0$ to $x=6$ (*id.*, 1893, p. 228) to nine places.

Ber and Bei, Ker and Kei Functions :—

These, which are really Bessel functions, are given by Savidge for the first thirty integral numbers (*Phil. Mag.*, xix., 1910). See also A. G. Webster, *Brit. Ass. Rep.*, 1912, p. 56.

Tables of Differential Equations of the 2nd, 3rd, and 4th order soluble in terms of Bessel's Functions :—

An exhaustive list, together with their solutions, is given in Jahnke and Emde, pp. 166–168.

TABLES OF VARIOUS INTEGRALS

$\int_0^x \frac{\sin x}{x} dx$ or $Si\ x$ and $\int_0^x \frac{\cos x}{x} dx$ or $Ci\ x$ have been calculated from $x=0$ to $x=1$ at intervals of $\cdot 01$ to eighteen places; from $x=1$ to $x=5$ at intervals of $\cdot 1$ to eleven places; from $x=5$ to $x=15$ (also for $x=20$) at intervals of unity to eleven places by Glaisher in *Phil. Trans.*, 1870, p. 367, also given in Jahnke-Emde for these values of the argument as well as for various greater values to four places.

$\int_{-\infty}^x \frac{e^x}{x} dx$ or $Ei\ x$: $Ei\ (\pm x)$ has been calculated by Glaisher to the same extent as $Si\ x$ and $Ci\ x$, and by Bretschneider in Grunert's *Archiv*, iii. p. 33, for $x=1, 2 \ldots 10$ to twenty places. $Ei\ x$, for $x=10, 11 \ldots 20$ is given to twenty places by J. P. Gram in *Publications of the Copenhagen Academy*, 1884, ii. No. 6, pp. 268–272.

Gram in some places extends Glaisher's table, giving $Ei\ x$ for $x=5\cdot 0$, $5\cdot 2, \ldots 20\cdot 0$ to eight, nine, or ten places. Jahnke-Emde has an extensive table to four places for $Ei\ x$ and $Ei\ (-x)$.

$\int_0^x \frac{dx}{\log x}$ or $li\ x$ calculated for $x=1000$, 10,000, 100,000, 200,000, . . . 600,000, and 1,000,000, by F. W. Bessel (see *Abhandlungen*, ii. p. 339). Calculated by J. von Soldner (Munich, 1809) from $x=0$ to $x=1$ at intervals of $\cdot 1$ to seven places, and thence at various intervals to 1220 to five or more places. Glaisher in his *Factor Tables*, § iii. (1883), gives $li\ x$ to nearest integer from 0 to 9,000,000 at intervals of 50,000.

$\int_0^x e^{-x^2} dx$ or $erf\ x$. J. F. Encke in *Berliner ast. Jahrbuch*, 1834, gives $\frac{2}{\sqrt{\pi}} erf\ x$ from $x=0$ to $x=2$ at intervals of $\cdot 01$ to seven places and $\frac{2}{\sqrt{\pi}}\ erf\ (\rho x)$ from $x=0$ to $x=3\cdot 4$ at intervals of $\cdot 01$, and thence to $x=5$ at intervals of $\cdot 1$ to five places ($\rho=\cdot 4769360$). Oppolzer in vol. ii. (1880) of *Lehrbuch zur Bahnbestimmung der Kometen und Planeten*, gives $erf\ x$ from 0 to 4·52 at intervals of ·01 to five places.

J. Burgess, *Trans. Roy. Soc. Edin.*, 1888, XXXIX. pt. ii., No. 9, gives very extensive tables of $\frac{2}{\sqrt{\pi}}\ erf\ x$. They extend from $x=0$ to 1·250 at intervals

of ·001 to nine places with 1st and 2nd differences, from $x=1$ to 3 at same intervals to fifteen places with 1st, 2nd, 3rd, and 4th differences, and from $x=3$ to $x=5$ at intervals of ·1 to fifteen places. Jahnke and Emde gives $\frac{2}{\sqrt{\pi}} = erf\,(x)$ from $x=0{\cdot}000$ to $x=1{\cdot}509$ and from $x=1{\cdot}50$ to $x=2{\cdot}89$ to four places. This book also gives to four places the first six derived functions defined by

$$\phi^{(p+1)}(x) = p\,!\,\frac{d}{dx}\left(\frac{2}{\sqrt{\pi}}erf\,x\right)\left\{(-1)^{p}\frac{(2x)^{p}}{0\,!\,p\,!} + (-1)^{p-1}\frac{(2x)^{p-2}}{1\,!\,(p-2)\,!} + \ . \ . \ . \right\}$$

from $x=0{\cdot}00$ to $x=3{\cdot}00$. They were calculated by H. Bruns (*Wahrscheinlichkeitsrechnung*, Leipzig, 1906, Teubner).

$\int_x^\infty e^{-x^2}dx$ or *erfc* (x) :—

H. Markoff, in *Table des valeurs de l'intégrale* $\int_x e^{-t^2}dt$ (St Petersburg, 1888), gives *erfc* x from $x=0$ to $x=4{\cdot}80$ at intervals of ·01 to eleven places with 1st, 2nd, and 3rd differences.

$Log_{10}\,(e^{x^2} erfc\ x)$:—

This is calculated by R. Radau in the "Annales de l'Observatoire de Paris" (*Mémoires*, 1888, xviii., B. 1–25), from $x=-0{\cdot}120$ to $x=1{\cdot}000$ to seven places. It is also given by C. Kramp, *Analyse des Réfractions* (Strasburg, 1798), from $x=0{\cdot}00$ to $x=3{\cdot}00$ to seven places. Also by F. W. Bessel, *Fundamenta Astronomiæ* (Koenigsberg, 1818) from $x=0{\cdot}00$ to $x=1{\cdot}00$ to seven places, together with the same for argument $\log_{10} x$ at intervals (in the argument) of ·01 between 0 and 1.

$\int_0^x e^{x^2}dx$. Given by H. G. Dawson from $x=0$ to $x=2$ to seven places in *Proc. Lond. Math. Soc.*, 1898, xxix. p. 521.

The Gamma Function :—

Legendre's calculation of $\log_{10}\Gamma(x)$ from $x=1{\cdot}000$ to $x=2{\cdot}000$ to twelve places with differences to the third order are printed by O. Schlömilch in *Analytische Studien* (1848), p. 183. A six-figure abridgment is given by B. Williamson, *Integral Calculus* (1884), p. 169.

$Log\ tan\left(\frac{\pi}{4}+\frac{\phi}{2}\right) = gd^{-1}(u)$:—

Given by C. Gudermann, *Theorie der potenzial- oder cyklisch-hyperbolischen Functionen* (Berlin, 1833), for every centesimal minute of the quadrant to seven places, and in particular from 88° to 100° to eleven places. A. M. Legendre, *Traité des fonctions elliptiques*, gives the same to twelve places for every half degree (sexagesimal). An extensive table is also given in Jahnke and Emde. The gudermannian is given in Becker and Van Orstrand to seven places from $u=0{\cdot}000$ to $u=3{\cdot}000$ radians and from $u=3{\cdot}00$ to $u=6{\cdot}00$ radians. This book also gives the antigudermannian to hundredths of a minute for every second in the quadrant. It should be consulted for information about more extensive tables.

Fresnel Integrals :—

$$C(x)=\int_0^x \cos \frac{1}{2}\pi x^2 dx=\frac{1}{\sqrt{2\pi}}\int_0^z \frac{\cos z}{\sqrt{z}}dz$$

$$S(x)=\int_0^x \sin \frac{1}{2}\pi x^2 dx=\frac{1}{\sqrt{2\pi}}\int_0^z \frac{\sin z}{\sqrt{z}}dz \quad \text{where } z=\frac{1}{2}\pi x^2.$$

$C(z)$ and $S(z)$ were given by Lommel, *Abh. Münch. Ak.* (2), 15,120 (1880), from $z=0$ to $z=50$ at unit intervals. From $z=0{\cdot}0$ to $z=50{\cdot}0$ at intervals of 0·1, and to four places, it is printed in Jahnke and Emde.

$C(x)$ and $S(x)$ are given at intervals of 0·1 from $x=0{\cdot}0$ to $x=5{\cdot}0$ by P. Gilbert (*Mém. cour. Acad. Bruxelles*, xxxi. 1863), and from $x=5{\cdot}1$ to $x=8{\cdot}5$ by W. Ignatowsky (*Ann. d. Physik* (4), xxiii. 894–898). To four places they are reproduced in Jahnke and Emde.

The Pearson Integral :—

$$F(r,n)=e^{-\frac{1}{2}\pi n}\int_0^\pi \sin^r x e^{nx} dx.$$

Log $F(r,\phi)$, where $n=r \tan \phi$, has been given by Lee, Yule, Cullis, and Pearson in *Brit. Assoc. Rep.* (1896, p. 70, and 1899, p. 65) for $n \leqslant r$, for successive integral values of r from 0 to 50 and for values of ϕ from 0° to 45° at intervals of 5°. A table to four places is printed in Jahnke and Emde.

For $G(r,n)=\int_0^\pi \sin^r x e^{nx} dx$, see *Brit. Assoc. Rep.*, 1896, p. 70, and 1899, p. 65.

Most of the above tables are exhibited. They are on loan chiefly from the Libraries of the Royal Society of Edinburgh; The Royal Observatory, Blackford Hill; and the University of Edinburgh.

IV. Notes on the Special Development of Calculating Ability.

By W. G. SMITH, M.A., Ph.D.

THE growth of calculating ability as it appears within the range of conditions in ordinary life is a matter of interest and importance. But when these conditions are absent, as they have been with not a few calculators, the interest is much heightened. Those who show distinguished ability in mental calculation may be young; they may owe little or nothing to education or to the stimulus of a cultured environment; they may even, while attaining a striking measure of success, be unable to read or write. The psychological problems which are involved are thrown into the clearest relief in the case of those who are quite young; but, while there are important observations on record in regard to such instances, the data which we possess refer, in the main and quite naturally, to the work of mental calculation as carried on in more mature years. We may, however, legitimately assume that whatever insight is gained with respect to the process of calculation in later years, may, with

appropriate qualifications, be applied in considering the problems of earlier development.[1]

That precocity is a marked characteristic of calculating ability is clear. Binet, referring to "the natural family of great calculators," estimates the age at which the ability appears as being on the average eight years.[2] A later writer, Mitchell, contends that it should be given as five to five and a half years.[3] Early development is a distinguishing feature of great men in science as in other provinces; but, as Binet remarks, the degree of precocity is perhaps nowhere so marked as it is in mental calculation.

Proceeding to consider the mental features which are presented in various forms by the calculators, and whose recognition may assist in understanding their achievements, we may note in the first place a deep interest in numbers. The presence of this characteristic might perhaps be assumed on general grounds. On the other hand, here, as at other points, it may be well to refer to observations relating directly to the work of calculators. In speaking of his own attainments, Bidder remarks that he has no particular turn of mind beyond a liking for figures, a liking which, he adds, many possess like him.[4] It was towards the age of six years, according to Binet, that Inaudi was seized by the passion for numbers. We learn that Rückle, whose gifts in the way of memory and calculation have been fully studied by Müller, possessed in his youth, and particularly in the period from the twelfth to the fourteenth year, a very intense interest in numbers, their analysis, and other features. It is easy to understand that such interest may form the stimulus to the persistent exercise of mental powers with respect to numbers, and to prolonged and cumulative practice. This result may, in addition, be favoured by the situation in which the boy is placed. Mondeux, Inaudi, and others were occupied in their early youth in tending sheep. Such an employment gives an opportunity for the development of this special form of talent. Even illness, or physical disability, may, as Mitchell points out, form a favourable condition, by preventing the boy from participating in ordinary games. It should, at the same time, be noted that numbers, while presenting certain abstract and universal features of experience, offer relatively simple relations for the work of calculation, and are capable of illustration in various simple forms. Mondeux is reported to have used pebbles in his calculations. Bidder used peas, marbles, and especially shot, in working out numerical relations. Other branches of study do not, it is clear, offer the same opportunities for early unaided progress.

It is obvious that intellectual activity is involved in the attainments of the calculator. But in attempting to formulate it as a definite factor in explanation, one is met by certain difficulties. The concept of this activity is not free from a certain indefiniteness, which we can hardly discuss here.

[1] The following works deal more or less comprehensively with the present subject:—"Arithmetical Prodigies," by E. W. Scripture, *American Journal of Psychology*, iv., 1891–92; *Les grand calculateurs et joueurs d'échecs*, by A. Binet, 1894; "Mathematical Prodigies," by F. D. Mitchell, *American Journal of Psychology*, xviii., 1907; *Zur Analyse der Gedächtnistätigkeit und des Vorstellungsverlaufes*, by G. E. Müller, 1911–13.

[2] *Op. cit.*, p. 191. [3] *Op. cit.*, p. 97.

[4] Here and elsewhere the reference is to G. P. Bidder, senior, whose paper on "Mental Calculation" appears in the *Proceedings of the Institute of Civil Engineers*, xv., 1855–56. His son, J. P. Bidder, will be referred to as Bidder, junior.

It may perhaps be taken as meaning generally the insight into the relations of objects. Now one is ordinarily not surprised to hear of instances where general intellectual ability is markedly present, but where mathematical ability is not conspicuous. But even the unreflective mind is struck by the circumstance that this calculating ability may be present in a comparatively isolated form. The case of Dase may be cited here. He showed a remarkable power of calculation, yet, as Schumacher writes to Gauss, it was impossible to get him to comprehend the first beginnings of mathematics.[1] We may suppose, in such a case as that of Dase, either an extremely one-sided form of intellectual ability, or a general ability which is limited by lack of interest in any other object in the mathematical sphere except that which is purely numerical. On both suppositions the matter involves difficulty.

Another aspect of this general problem is presented by the case, studied with great care by Wizel,[2] of a woman possessing considerable power of calculation who is in an imbecile condition, the result of an attack of typhus in her seventh year. Her mental life is characterised by alternation between a state of indifference or apathy and one of excitement; apart from arithmetical knowledge, the range of ideas is very limited, the poverty in abstract and general ideas being specially noticeable; her power of judgment is on the level of that of a child of three years. Her abilities in calculation are described by Wizel as follows:—"Apart from addition and subtraction, which she performs slowly and often incorrectly, she manifests unusual abilities in the sphere of multiplication, and partly also in that of division. In spite of her retarded intelligence she carries on these operations rapidly, and, what is more remarkable, with much greater rapidity than an ordinary normal individual." With three-place numbers the results are much poorer: "she multiplies in memory three-place by one-place numbers pretty well, but here her calculating abilities terminate." Wizel suggests that where the problems are solved immediately the memory alone is exercised: where several seconds are required, definite methods, *e.g.* factorising, are employed. Those problems in which the number 16 is involved are solved with special facility, apparently because in earlier years the patient was very fond of collecting objects, *e.g.* coins, which were arranged and counted in groups of sixteen. Referring to the main lines of explanation already mentioned, one may suppose that the phenomena are due to lack of interest in almost everything except numbers,[3] or to the survival, in the midst of extensive pathological impairment, of memory for numbers and the related calculating ability. It is reasonable to consider that both factors are at work, the latter being the more important. It may be noted that the preservation of abilities connected with number is a feature in certain cases of aphasia.

The importance of memory has been justly emphasised in the discussion of the present topic. Its function may be considered here in two main

[1] *Briefwechsel zwischen C. F. Gauss und H. C. Schumacher*, v. S. 295.

[2] "Ein Fall von phänomenalen Rechentalent bei einem Imbecillen," *Archiv für Psychiatrie*, xxxviii., 1904.

[3] It is remarked by Wizel (*op. cit.*, S. 128), with reference to the two topics—her supposed persecutions, and calculation:—"Round these subjects the conversation usually turns. Otherwise she is interested in nothing, speaks of nothing, busies herself with nothing."

respects, which in practice are inextricably bound together. In the first place, it enables the calculator to keep in permanent possession those properties of numbers which he has already grasped; in the second place, it enables him to retain the actual data, the particular products, or other features of the special problem before him at the moment. These are analogous, on the one hand, to the general memory of a language, and, on the other, to the knowledge at any moment of what has been said at the preceding stages of a conversation. As an illustration of the former, there may be noted the fact, mentioned by Cauchy in the report to the Académie des Sciences,[1] that Mondeux knew almost by heart the squares of all the whole numbers up to 100. Bidder points out the importance of knowing by heart such facts as the number of seconds in a year or the number of inches in a mile. Rückle, in his twelfth year, knew by heart as regards all the numbers up to 1000 whether they were primes or not, and in the latter case what their factors were. As regards the second direction in which memory is active, it may be noted that, according to Bidder, the key to mental calculation lies in registering only one fact at a time, the strain in calculation being due to this work of registration. Thus in a complex multiplication he goes through a series of operations, "the last result in each operation being alone registered by the memory, all the previous results being consecutively obliterated until a total product is obtained";[2] what is thus not kept in view can be recollected when needed. It may be mentioned that, according to Binet, Inaudi could recall at the close of a public exhibition 300 figures involved in the different problems he had dealt with, and, after the lapse of sixteen to eighteen hours, many of the numbers used on the previous evening, though but few of those of the preceding evenings.

With regard to the ability to learn by heart and reproduce immediately a series of numbers, the following data given among others by Binet and Müller are of interest. In order to learn by heart a series of 105 digits read aloud and thereafter to repeat it, Inaudi required twelve minutes; to learn a written series of 100 digits and write it out, Diamandi required twenty-five minutes; to learn a visual series of 102 digits and repeat it by heart, Rückle required, on the average, approximately five minutes and forty seconds; Arnould, using special mnemo-technical devices, required, when tested in the same way as Diamandi, fifteen minutes.

In the investigations described by Binet,[3] a fact of considerable importance was brought out, viz. that Inaudi's imagery is of the auditory, or auditory-motor type, not of the visual type. Rückle's type, on the other hand, is visual, though, when it is advantageous, he can use auditory-motor factors. The fact referred to above is significant in relation to the view that mental calculation is carried out on a visual basis. Thus Bidder junior says:[4]—"If I perform a sum mentally, it always proceeds in a visible form in my mind; indeed, I can conceive no other way possible of doing mental arithmetic." An attempt has been made by Proctor[5] on the basis of his

[1] *Comptes rendus*, xi., 1840, p. 953. [2] *Op. cit.*, p. 260. [3] *Op. cit.*, ch. v.; *cf.* J. M. Charcot, *Comptes rendus*, cxiv., 1872. [4] *Spectator*, li., 1878, p. 1634.

[5] *Cornhill Magazine*, xxxii., 1875; this article is reprinted in "Science Byways," *Belgravia*, xxxviii., 1879.

own early experiences to explain mental calculation by a special form of visual imagery. It was suggested that while the reference to ordinary arithmetical processes was inadequate, it was possible to find an explanation in the possession of an enhanced power to picture numbers as an assemblage of spots, or dots, arranged in columns which could be modified with the utmost facility and whose relations could be immediately grasped. At one time Proctor considered this to be the general method; later on he admitted that its use was limited, while contending that it gave the best account of Colburn's feats. The general value of the suggestion may be acknowleged; the recorded observations do not, however, support the view that this method of "mental marshalling" has actually been employed by great calculators. In connection with the visual type, attention may be called to the occasional presence of number forms in which the figures appear in a definite spatial order. It is of interest that Galton,[1] who first studied this topic, had the existence of such forms brought to his notice by Bidder, junior, who inherited, in some measure, his father's gift of calculation. Such a form was detected by Binet in the case of Diamandi. It has been urged by Hennig that "the possessors of number diagrams in general not only have a better memory for numbers, but also are apt to be much better in mental calculation" than those who lack this feature.[2] A wider review of the facts leads, however, to the conclusion that an unqualified assertion of the advantages of this feature cannot reasonably be made.[3]

Whilst with certain calculators the memory for numbers is merely one phase of a general power which shows itself in other directions also, in other cases the memory, while excellent as regards numbers, is relatively poor in other directions. In illustration of the latter case, it may be noted that Mondeux has much difficulty in retaining a name or an address. Inaudi's memory, again, is not in any way remarkable beyond the sphere of number. In attempting to understand this feature, suggestions have been made of an innate mental ability or a special development of memory due to certain assignable conditions. The latter suggestion seems to supply the basis for an adequate explanation of the phenomena. It has been shown by experimental methods that memory can be trained in such a way as to improve markedly its effectiveness both in immediate reproduction and in its more permanent phases. It is not necessary to discuss the question whether such a result is to be interpreted as a real change in power, or as being due to the more efficient and economical use of powers already possessed. Assuming that the latter view is the more probable, one may refer to general conditions which have been recognised, such as the increased ability to concentrate and maintain attention, the diminution of fatigue, and better adjustment of emotional and active tendencies to the work which is being prosecuted. It will be acknowledged that the interest in numbers, to which reference has already been made, forms, when it is persistent, a powerful motive to improvement of memory in this sphere. The admiration readily accorded to unusual attainments

[1] *Inquiries into Human Faculty and Development*, 1883 (Number Forms).

[2] "Entstehung und Bedeutung der Synopsien," *Zeitschrift für Psycholcgie und Physiologie der Sinnesorgane*, x., 1896, S. 215.

[3] *Cf.* Müller, *op. cit.*, Abschnitt 8, Kap. 3.

in the direction of mental calculation will inevitably assist in reinforcing the original interest. And it is to be observed that growing excellence in this sphere may be accompanied by an actual lessening of power in other subjects, especially in cases where there may be recognised a certain narrowness or poverty of mental content.

Reference may next be made to the influence of attention. In summarising the characteristics of Rückle, who combines a remarkable memory and a high calculating ability with mathematical culture, Müller points out that he possesses in a high degree the power of concentrating his attention with full intensity, and that after a few introductory words he is ready to devote his full strength to each problem put before him, regardless of movement or experimental preparations in the room where he is working. One may note a similar attitude in the calculator Inaudi, who is not troubled by the noise or conversation going on around him on the stage, and in the case of Mondeux, of whom it is stated, that when his attention is directed to the numbers which have to be combined, his thought can follow the problems "as if he were completely isolated from all that surrounds him." What may be regarded as another aspect of attention is indicated by certain remarks which Schumacher makes regarding Dase. Thus in one instance he writes:[1] "His rapid knowledge of numbers is to me almost the most remarkable thing. If you throw down a handful of peas, the most cursory glance enables him to tell their number." A similar remark is made regarding his ability to grasp a line of figures. We seem to have before us in such facts that feature of attention by which the calculator is able to grasp with the utmost rapidity, and almost in a single act, the significance of a complex group of figures or other connected data which are presented to him. The advance which the child makes in passing from the reading of letters to the unitary grasp of words and higher complexes is made by the skilled calculator in his handling of higher numerical groups. We may say generally of attention that, with regard to memory, it develops concentration on the relevant features of the subject-matter which is presented, and facilitates the learning process by which knowledge of specific relations is built up, as well as the subsequent process in which this knowledge is recalled, while, with regard to intellectual activity, it brings the problem vividly before the calculator, and enables him to grasp the complex relations of what is presented with the utmost rapidity.[2]

A special feature of the calculating process is perhaps to be found in the case of Colburn. To the inquiries made regarding the methods employed in his calculations, he was for some time unable to give any answer, though evidently trying honestly to enlighten his friends. His account of the discovery, in his tenth year, of the method of factorising, which, for upwards of three years, he had been unable to give, may be quoted.[3] "It was on the night of 17th December 1813, while in the City of Edinburgh, that he waked up, and, speaking to his father, said, 'I can tell you how I find out the factors.' His father rose, obtained a light, and, beginning to write, took down a brief sketch, from which the rule was described and the following tables formed."

[1] *Op. cit.*, S. 296.

[2] Reference may be made to experimental investigations regarding the range, or span, of attention and of consciousness; *cf.* W. Wundt, *Grundzüge der physiol. Psychologie*, iii.

[3] *A Memoir of Zerah Colburn written by himself*, 1833, p. 183.

He then proceeds to give a set of tables of the various pairs of factors which, multiplied, give the two-place endings up to 99. Referring to his backwardness in giving explanations several years earlier, he remarks that it was not owing to ignorance of the methods he pursued; "he rather thinks it was on account of a certain weakness of the mind which prevented him from taking at once such a general and comprehensive view of the subject, as to reduce his ideas to a regular system in examination." This explanation is very reasonable, but it may be suggested that it hardly seems to give an adequate account of the suddenness of the discovery in the instance cited above, or in another when he was at dinner with a friend, and, as we learn,—"Suddenly Zerah said he thought he could tell how he extracted roots." The question then may be raised whether such observations do not point to a certain ability to carry on the operations of calculation in a mental region, which, to speak figuratively, is beyond the margin of attentive processes, or is subconscious, if we may introduce this ambiguous term.[1] When the complexity of a cognitive activity is considerable and the rapidity with which it is carried out is great, we may be readily aware of its results, and yet may find it difficult, with even special training, to give a full account of the character of the processes involved. Something of this kind is probably present in rapid expert calculation, and we may perhaps fairly suppose that Colburn's observations indicate a process of this kind. If not carried out originally in this form, his calculations may, owing to some special circumstance, have readily passed into this form in the course of persistent exercise.[2]

Having thus reviewed the chief features of the mental processes involved in the actual work of great calculators, we may turn to the problem of the speed of their activity. There are two sides of this problem—the arithmetical and the psychological. As regards the first, attention may be called to the circumstance that, in the course of their persistent occupation with numbers, the calculators have in fact discovered various procedures and various properties of numbers, by which problems can be solved with greatly increased facility. One instance will suffice—the discovery by Colburn of the significance, with respect to the finding of roots, of two-figure endings together with the first one, two, or more figures in a lengthy number of five, six, or more digits.

Passing to the psychological problem, we may refer again to the calculator's knowledge of many properties of numbers which he possesses permanently by memory, and in respect of which the labour and time of calculation are

[1] An observation made by Gauss (*op. cit.*, S. 297) may be cited in this connection. After referring to the great psychological interest which an adequate analysis of Dase's mental processes in calculation would possess, and to the difficulties involved, he proceeds:—"For, indeed, I have had many experiences of my own, which remain puzzling to me. The following is an instance. Sometimes, while I walk along a certain path, I begin in thought to count the steps . . . thus I count on to 100, and then begin again. When, however, this is once started, it is all done unconsciously; I think about quite different things, notice attentively anything remarkable—only I have to avoid speaking meanwhile—and after some time I begin to be aware that I am continuing to count in time."

[2] It may be noted here that according to Mitchell (*op. cit.*, pp. 100 ff.) three grades of ability may be distinguished in the great calculators. In the first the operation is one of pure counting, and it is the properties of numbers and series that are thought of, while in the second the interest relates principally to the operations of calculation. In the third real mathematical ability is found. Mental arithmetic grows naturally and independently out of counting.

spared. "It is certain," Binet remarks, "that M. Inaudi knows in advance many of the results of partial calculations which he utilises on each new occasion; his memory has retained the roots of a great number of perfect squares; he knows also the number of hours, minutes, and seconds in the year, the month, and the day." The constant practice carried on by the calculators increases, in addition, the facility with which the appropriate data are recalled. A remark by Bidder is interesting in this connection:—"Whenever, as in calculation, I feel called upon to make use of the stores of my mind, they seem to rise with the rapidity of lightning." Further, continual exercise will enable the intellectual activity, especially in its relations with attention, to be carried on with growing rapidity.[1] One must at the same time keep in mind that there are certain innate, unacquired differences between individuals in the rapidity with which mental activities are carried on.

Scripture calls attention[2] to the great shortening in time which may be attained "if the adding, subtracting, multiplying, etc., can be done before the numbers themselves come into full consciousness," and if all superfluous processes are omitted. He suggests also that this feature may explain Colburn's inability to explain his methods. Binet points out[3] that Inaudi, to whom the subject-matter of a problem is read aloud, begins to calculate while listening to the series of data, and that Diamandi, in the process of learning a series of numbers, does not keep separate the processes of reading, learning, and of the final writing out, the processes being in reality *enchevêtrées*. Such a union of processes may possibly exist in other cases also, and, if so, it would help to explain the rapidity with which the results are reached. Reference may be made in this connection to observations which indicate the ability to carry on at the same time two distinct series of mental operations. Binet remarks[4] with regard to Inaudi:—"We have seen him sustain a conversation with M. Charcot at the Salpêtrière while he solved mentally a complicated problem; this conversation did not confuse him in his calculations, it simply prolonged their duration." We are told of Buxton,[5] the Derbyshire labourer, whose calculations were not remarkable for their rapidity, that "he would suffer two people to propose different questions, one immediately after the other, and give each his respective answer without the least confusion"; and, again, that "he will talk with you freely whilst he is doing his questions, it being no molestation to him, but enough to confound a penman." It is, of course, clear that such observations do not give a rigid proof of the complete concurrence of the different series of activities. A remark of Bidder, junior, emphasises the importance of the self-reliance which prolonged practice secures:—"I am certain that unhesitating confidence is half the battle. In mental arithmetic it is most

[1] The following remarks by Bidder, junior (*op. cit.*, p. 634), refer to his father's powers. "The second faculty, that of rapid operation, was no doubt congenital, but developed by incessant practice and by the confidence thereby acquired. . . When I speak of 'incessant practice,' I do not mean deliberate drilling of set purpose; but with my father, as with myself, the mental handling of numbers or playing with figures afforded a positive pleasure and constant occupation of leisure moments."

[2] *Op. cit.*, p. 44. [3] *Op. cit.*, pp. 80, 123. [4] *Op. cit.*, p. 37.

[5] *Gentleman's Magazine*, xxi., 1751, pp. 61, 347.

true that he who hesitates is lost." Müller mentions an observation of Rückle regarding multiplication, to the effect that one must above all avoid hesitation between different modes of procedure, and that one learns by practice to know at once what method to adopt.

While the various processes indicated in the earlier and later sections of this brief review have of necessity been discussed separately, it is not meant to be suggested that they exist in isolation. The various features are no doubt intimately connected in actual practice. At the same time we have to keep in mind that different individuals may reach a similar result as regards the solution of problems through complex activities which may differ both in their constituent processes and in the varying prominence which certain common elements possess. Recognising the co-operation of factors and the differentiation of the complex activities, we seem to reach a reasonably adequate understanding of the achievements in calculation both in early life and in more mature years.

The following is a list of Supplementary Exhibits in Section C :—

(1) **Some Extraordinary Examples in Mental Calculations, including a Sum of Nine Figures multiplied by the same Number,** performed at the Bank of England, by G. Bidder, a Devonshire youth, not thirteen years of age. Thirty-six pages. 6·9″×4·3″.

Lent by Mrs Blackmore.

(2) One Volume of the **Su-li Ching-yün,** the "Imperial Treatise on Mathematics," prepared under the patronage of the Emperor Kang-he (1662–1722). This appeared at Peking in 1713, and consists of fifty-three books, treating chiefly of European mathematics. The work was a compilation, but no names of authors or contributors are given. This volume shows part of the great table of logarithms, to ten decimal places, based on the Vlacq tables of 1628.

This was part of the *Ku kin tu shu tseih ching*, "Complete Collection of Ancient and Modern Books," the great encyclopædia in 6109 volumes.

Lent by Professor David Eugene Smith.

(3) The **Tai shin Rio su hio,** a manuscript possibly written by one *Do bin wun foo* (name on the seal on the first page). In it appears a table of logarithms to seven decimal places. The date is probably about 1800.

Lent by Professor David Eugene Smith.

(4) **The Braille Tables of Logarithms and Figures,** as embodied in Eggar's *Mechanics*, prepared for the use of the blind.

Lent by Henry Stainsby.

PORTRAIT OF CHARLES BABBAGE.

[To face p. 124.

Section D

CALCULATING MACHINES

Calculating Machines. By F. J. W. Whipple, M.A.

The following notes on calculating machines are on the lines of the *Catalogue raisonnée* which I prepared for the Exhibition in connection with the Fifth International Congress of Mathematicians held at Cambridge in 1912. The blocks are lent by the Cambridge University Press. I wish to make it clear that my point of view is that of the user of a machine who wishes to have a general idea of how it works rather than that of the expert who has to master every detail. I propose to confine my remarks to purely arithmetical machines, and say nothing of other apparatus, such as slide-rules or mechanical integrators.

It is convenient, in discussing arithmetical calculating machines, to take the fundamental operations of arithmetic in the following order:—numeration, addition, multiplication, subtraction, and division. For mere numeration or counting, there are two systems in general use. The simpler to construct is the one in which the wheels, whose position indicates the values of various digits, are always in gear with one another, as in an ordinary clock, and the figures of each denomination change gradually. When we look at a clock which shows twenty-eight seconds after eighteen minutes past three, we really see the hands indicating 3 hours + about $\frac{1}{3}$, 18 minutes + about $\frac{1}{2}$, and 28 seconds, respectively. For time, this is the most satisfactory system, but for most purposes it is easier to read figures presented to the eye as they would be written down. It is important to notice, however, that in a counter which shows figures in this way, the wheels cannot be continually in gear with one another. An example which shows the advantage of the displayed digit system is furnished by the cup anemometer.

Addition.—The process of addition involves two distinct operations, the addition of digits and the carrying of figures from one denomination to the next.

As far as I am aware, there is no machine which can be said to know the addition table. If 5 is shown on a counter and 3 has to be added to it, then the operation of adding 1 is gone through three times in rapid succession; there is not a sudden jump from 5 to 8.

Methods of Adding a Digit

The devices used for ensuring the addition of a particular digit determined by the operator may be classified as follows :—1, rocking segments ; 2, stepped reckoners ; 3, alternative racks ; 4, variable cog wheels.

1. *The rocking segment* is shown in fig. 1. Whilst the segment is turning in the direction shown by the arrow it is in gear with the counter C. When turning back again it is thrown out of gear. The angle through which the rocking segment can turn is settled by the key which has been pressed down (7 in the diagram).

The rocking segment will be found in the Cash Register and in Burrough's Adding Machine. In these machines the segment is turned by means of a

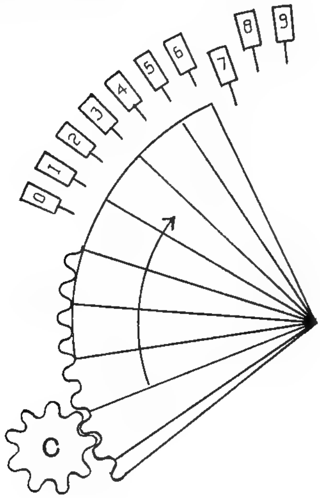


Fig. 1.

handle or by electric power. In the Comptometer the pressing of the key not only decides the range of the rocking segment, but causes it to rock.

2. *The Stepped Reckoner.*—The wheel R in fig. 2 is stepped, *i.e.* the cogs do not cover its entire length, but some are longer than others. When the wheel J is in the position shown in the diagram, only one of the cogs on R can engage with one on J. If, however, J were moved to the right until the pointer was under the 3, then three of the cogs on R would engage. Thus one turn of R will be recorded by a 1 or 3 on the counter, as the case may be. The stepped reckoner is used for addition in machines of the Thomas type, examples of which are the Arithmometer,[1] the Saxonia, and the T.I.M. The drawback of the system is the slow method of adjusting the sliding piece J. In a machine used especially for adding, the slide would have to be set by pressing a key.[2]

3. In the Mercedes machines the cog wheel J is adjusted in the same way,

[1] The Arithmometer is of British manufacture, and is notable for the smoothness of its action.

[2] This is done in the XxX machine (*Zeitschrift für Vermessungswissen*, 1913, S. 716).

but instead of stepped reckoners there are *racks* which move through different amplitudes. A single set of racks suffices to turn all the counters.

4. *Wheel with a Variable Number of Cogs.*—By means of the handle H the ring R is pushed through slots in the sliding knobs K. The wheel in the diagram has five knobs; by moving the handle H clockwise the number of knobs can

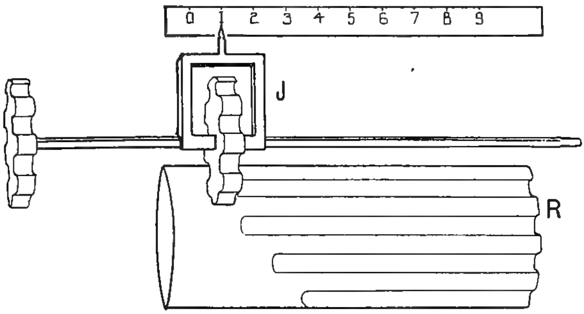


FIG. 2.

be increased to six. When the handle H has been adjusted the wheel is turned as a whole, and the knobs K knock the counter as they pass it.

This neat device is found in the popular Brunsviga machine.

Carrying.—The mechanism in an adding machine undertakes a task which is beyond the human brain. If a man has to add together two numbers such as 526314 and 131524, he has to think of the additions of separate

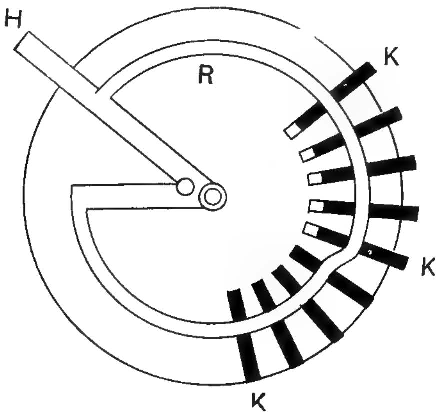


FIG. 3.

orders of magnitude *seriatim*: as a general rule the machine can attend to all the additions simultaneously. If the counter of the machine is watched while the handle is turned slowly, the digits are seen to change gradually but independently. On the other hand, when carrying has to be dealt with, the operation on the units column must be timed to precede the operation in the tens column, and so on. When unity is added to 995999, the transformation must begin on the right and stop short at the fourth figure. It cannot begin everywhere simultaneously.

It will be seen that carried figures may arise in two ways, which the designer

of a calculating machine must regard as distinct. If to 57447 the number 21586 is added, then, apart from the carried figures, the sum is 78923. Carried *ones* are now waiting to be added to the 2 and to the 9. It is not until after these *ones* have been added that the *one* which is to be added to the 8 appears.

The mechanism which is used for controlling the carrying of figures is the most delicate part of a calculating machine. The details, which vary in the different types, are not easy to explain without models.

Multiplication.—Multiplication is essentially repeated addition, and therefore any adding machine can be used for multiplication, at any rate when small multipliers are concerned. For such work the comptometer will be found most useful. For dealing with large multipliers, some method of changing the place value of figures by sliding the part of the apparatus carrying the multiplicand relative to the part carrying the partial product is essential.

It should be noted that it is practically impossible to deal with English coinage, weights and measures, without expressing them in the decimal system, thus it is customary to express shillings and pence as decimals of a pound. This can be done with a calculating machine with less risk of error than in ordinary arithmetic, as there is less temptation to round off the figures and retain too few decimal places.

As we have already remarked, multiplication is repeated addition, and the ordinary multiplying machine goes through the process of addition: to multiply by 7, the adding process must be repeated seven times, as seven times the multiplicand has to be added to zero. The Millionaire calculating machine differs from the others in that it contains an automatic multiplication table. A marker is set, say to 4, and a pointer to 7, and the product 28 is recorded after a single turn of the handle. During this turn there are two distinct operations: at the end of the first half-turn the 2 appears in the right place in the product and the product-carriage moves one place to the left: in the second half-turn the 8 appears to the right of the 2. This effect is secured by controlling the amplitude of the motion of racks which move under pinions similar to those used with the stepped reckoner (fig. 2). Corresponding to each multiplier there is a tongue-plate which forms a multiplication table. For example, the "7" tongue-plate has nine pairs of tongues, the lengths of which correspond in length to so many cogs on the racks, 0, 7; 1, 4; 2, 1; 2, 8; 3, 5; etc. When 4 is multiplied by 7 the fourth rack is pushed by a short tongue on the seventh tongue-piece through two teeth, then the tongue-piece is itself displaced laterally, whilst the rack returns to its original position, and finally a longer tongue pushes the same rack through eight teeth.

Subtraction.—The process of subtraction being the reverse of addition, it might be expected that any adding machine might be used for subtraction by reversing the motion of the handle. This would lead to difficulties, however, as the process of carrying tens must run from right to left in subtraction as well as in addition. Accordingly, it is usual to have a switch which reverses the motion of the main shaft whilst keeping the same direction of rotation of the handle.

In some machines there is no separate mechanism for subtraction, but the computer adds 999356 when he wishes to subtract 000644.

Division.—The process of division with a calculating machine is closely analogous with ordinary long division. The computer has to be very alert, or he makes his quotient too big and has to retrace his steps. For many calculations it is advisable to use a table of reciprocals, and substitute multiplication for division.

There is one machine, however, the Mercedes-Euklid,[1] which is especially designed for division. The method adopted may be described as successive approximation to the quotient from above and below.

As a simple illustration let us consider the division of 10 by 7. The first process is subtraction, which is effected in machines of this type by the addition of the complementary number; to subtract 7, the machine adds 3, 93, or 993, as the case may be, according to the place value. Now if 93 is added to 10 twice, the sum is 196. So after two additions 2 appears as the first approximation to the quotient and 96 is the corresponding "remainder." The mechanism prevents the handle from being turned further. The operator is warned thereby that this stage of the process is complete: he moves a pair of keys; the carriage shifts to change the place value of the divisor, and the handle is set free for the next step in the division.

	10
	93
1	(1)03
	93
2	96
	7
19	967
	7
18	974
	7
17	981
	7
16	988
	7
15	995
	7
14	(1)002

During this stage the quotient 2·0, which is too great, is reduced. At each turn of the handle the quotient is reduced by a unit in the second place, and at the same time the remainder is increased by 7 in the corresponding place. As long as the 7's can be added without any 10 being carried on the left of the sum, the handle turns freely.

Now, starting from 960, and adding successive 7's, we arrive after six additions at 1002; the figures 002 appear as the remainder, and as the 1 cannot be "carried," the handle locks again. The quotient is now 20−6 or 14, and the remainder 2, *i.e.* at this stage we have the same approximation as in ordinary arithmetic and a quotient which is too small. The next step gives too big a quotient, and so on.

[1] *Zeitschrift für Instrumentenkunde*, 1910.

Successive remainders and quotients are (ignoring the decimal point):

96	002	9999	00004	999998
2	14	143	1428	14286

These correspond to the equations

$$\frac{10}{7}=2+\frac{96-100}{7}$$

$$\frac{100}{7}=14+\frac{2}{7}$$

$$\frac{1000}{7}=143+\frac{9999-10000}{7}$$

$$\frac{10000}{7}=1428+\frac{4}{7}$$

and

$$\frac{100000}{7}=14286+\frac{999998-1000000}{7}.$$

Two features of this machine may be mentioned as displaying remarkable ingenuity—the way of determining the complement of a number and the system according to which the handle is stopped at the right place during division.

If we want to write down the complement of any number such as 374093, we write down the difference between each figure and 9 with one exception, viz., we must take the difference between the last figure and 10. How can this exception be allowed for without depriving the machine of all symmetry? The answer to this question has been found in the provision of a hidden extra digit on the right. This digit is always zero for addition and 10 for subtraction. Thus if we write t for the digit 10, we may say that the machine takes $625906 \cdot t$ as the complement of $374093 \cdot 0$.

It will be remembered that in the course of a division operation the locking of the crank is the end of each step. The locking in addition is a simple enough process. If 041 is added to 095, the first half-turn brings 036 on to the counter, and in the next half-turn the carried 1 appears, making 136. If, however, 41 is added to 95, the first half-turn brings 36 on to the counter, and in the next half-turn the locking catch slips into position. When the machine is adjusted for subtraction, the actual process is the addition of the complementary number. Thus, in the case discussed above as an example of division, 93 is being added to 10: the first sum is (1)03, but the carrying of the 1 does not lock the crank: the second sum is only 96, and it is necessary for the process to stop at this stage. Accordingly, we have the contrast: in addition the occurrence of the 1 to carry locks the crank; in subtraction the lack of the 1 to carry locks it.

The Scope for Improvement of Calculating Machines

There are certain developments in calculating machines which would be of considerable value, and which could be made if there were sufficient demand. In the first place, it is remarkable that no multiplying machine

which does long multiplication automatically is on the market at present. With such a machine it would be possible to set up the multiplier and multiplicand and then turn the handle without giving it any conscious attention until the locking of the motion showed that the operation was complete and the product was ready to be read off. I fancy that it would not be difficult to modify the Thomas machine to enable it to act in this way.

A more valuable invention would be a multiplying machine which could do continued multiplication. If three or more numbers have to be multiplied together, the first product has to be used as one of the factors for obtaining the second product. The transfer of the figures from one set of indicators to another is likely to lead to mistakes, and in any case wastes time. In such problems as the formation of a compound interest table or the calculation term by term of a hypergeometric series, the additional labour is so irksome that the computer would probably prefer to use logarithms.

Two ways of making a machine which would overcome the difficulty occur to one. There might be two indicators related in such a way that either could stand for multiplicand or for product; or, again, there might be three indicators, A, B, C, mounted on a cylinder, so that when A was used for the multiplicand the product appeared on B; when B was the multiplicand, the product was on C; and finally when C was multiplicand, the product was on A.

The mechanical difficulties in making continued product machines would be considerable, but by no means insuperable.

Finally, I should like to raise the question whether there is sufficient scope for a machine for calculating tables to justify its construction. Large sums of public money were voted in the early nineteenth century for the construction of Babbage's Difference Engine, which was to be used for this purpose. In these days of automatic tools, Babbage's Engine could be constructed at a moderate cost, but it would probably be better to start afresh and re-design it throughout. The story of Babbage's efforts end at present in a confession of national failure, and it would be gratifying to British mathematicians if a happier sequel could be written in our annals. Will the potential importance of the Difference Engine as a tool in the computer's workshop be recognised again, or shall we have to admit that Babbage's invention was never brought to perfection because the need for it was imaginary?

Exhibit of machines from the Mathematical Laboratory, University of Edinburgh:—

Archimedes.
Brunsviga (ordinary and miniature).
Burroughs Adding (printing).
Comptometer (two).
Mercedes-Euklid.
Millionaire.
Tate's Arithmometer.

All the machines described in Section D are exhibited and demonstrated.

I. Calculating Machines Described and Exhibited

(1) The "Archimedes" Calculating Machine

The Glashütter calculating machine "Archimedes" brings a new model into the market. The endeavour of every manufacturer of calculating machines is to reduce their size and weight without detriment to their stability

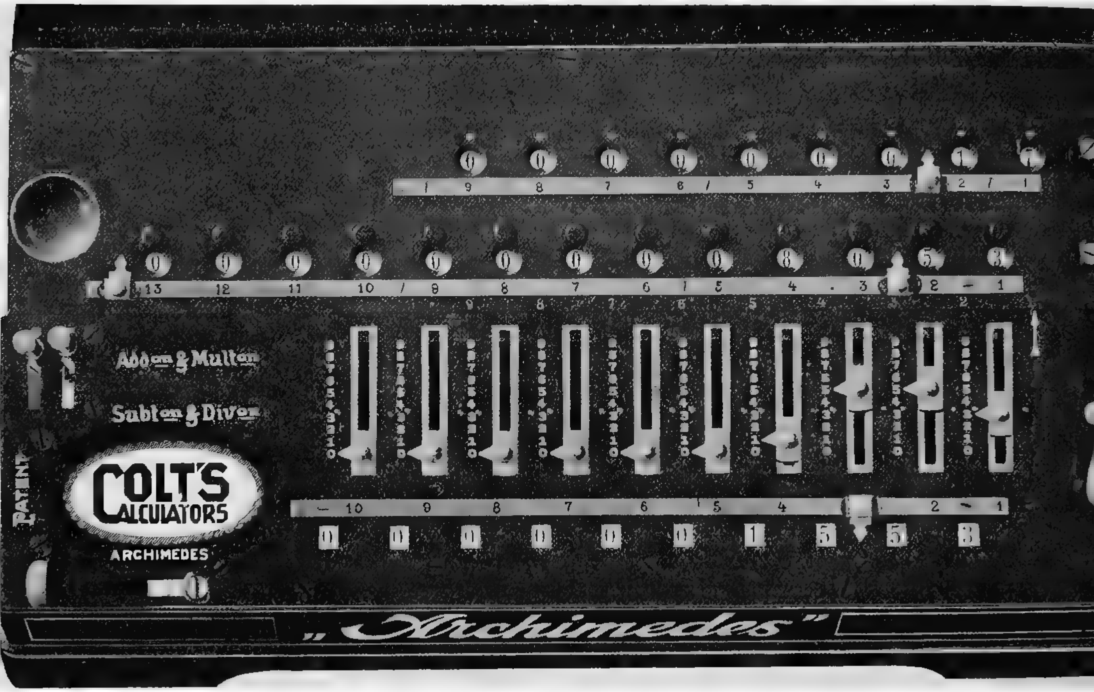


FIG. 1.

and efficiency. The new Glashütter calculating machine "Archimedes" weighs only 7 kg., and works extremely smoothly and silently.

In the accompanying diagram (fig. 2) the essential parts of the setting and the counting mechanism of the "Archimedes" are shown. First of all, in the right-hand bottom corner is the stepped reckoner, invented originally by Leibnitz. It is a cylinder, on the outer surface of which nine teeth of increasing length are so arranged that they occupy about one-fourth of the circumference. For each place in the setting mechanism a similar cylinder (1) is provided and set on a square axle. All the axles are driven from the shaft (3) by a crank-handle, by means of pairs of bevel wheels (2). Corresponding to the turning of the crank in a positive direction, the stepped cylinders turn so that the tooth corresponding to the digit *one* is the last to come into gear. Above these cylinders, and close to the covering plate, there is a square axle, on which is a sliding pinion (4) with ten teeth, which engages with the teeth in the cylinder. Each pinion is gripped by a fork-shaped continuation of the sliding indicator on the setting plate above, and moves simul-

taneously with it. It is thus obvious that the pinion, from the position it has received through the setting of the index, is rotated, when the cylinder is caused to revolve, by as many teeth as the cylinder bears in the plane corresponding to the digit set. The same amount of rotation is also received by the square axle which carries the pinion, and with it the pair of bevel wheels (7), which slide likewise on this axle. By means of this sliding it is now possible to transfer the rotary motion of the square axle in either the one or the other direction to the vertical axle (8), which bears at its upper end the figure disc. In the position represented in the diagram the figure disc will turn in a positive direction, *i.e.* the digits will appear in an ascending series at an indicator hole situated above it. If, however, the bevel wheels slide so that the other one engages the vertical shaft, the numbers will appear in a descending series. In each case, in the transition from 9 to 0 or from 0 to 9, the axle (8) will make a complete revolution, and the finger attached to it (9)

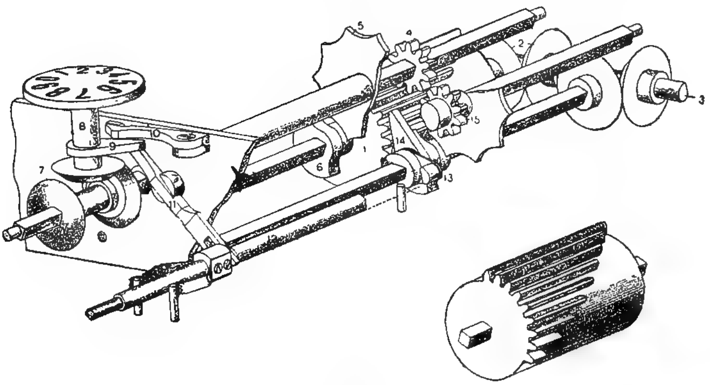


FIG. 2.

will press the nose-shaped end of the lever (10) backwards. The lever (10) operates in turn on one end of the lever (11), which is pivoted in the middle, the lower end of which is fork-shaped and fits with this fork into a notch in the sliding rod (12). The latter is kept in whichever position it may take up by springs for the purpose, and has at the rear end a fork which adjusts, according to the movement of the rod (12), the single tooth (14). This slides on the square axle of the stepped cylinder in the adjacent place. In the normal position, that is, so long as there is no contact between (9) and (10), the plane in which the single (14) tooth turns is behind the plane of the pinion which is fixed on the "setting" axle of the place immediately above, so that when it turns no engagement with this wheel results. But if the rod (12) is pushed forward, the tooth (14) will in turning engage with the teeth of (15), and thus turn the "setting" axle of the place immediately above one-tenth further round, which results in the raising or lowering of the following place by a unit, as the case may be.

Besides these parts, which are absolutely necessary for the counting and carrying, there must also be provided other contrivances to destroy the momentum of the rotating parts when the handle is turned quickly. This safeguard is carefully executed in the "Archimedes." There are also safe-

guards which prevent a displacement of the reversing lever, when the crank is not at rest.

The axles (8), which carry the figure discs, are not situated together with the other parts immediately in the body of the machine, but under a hinged plate or carriage (fig. 3), which may be lifted up and which may be slid along its axis. By sliding the plate from place to place in the row, the axles of the

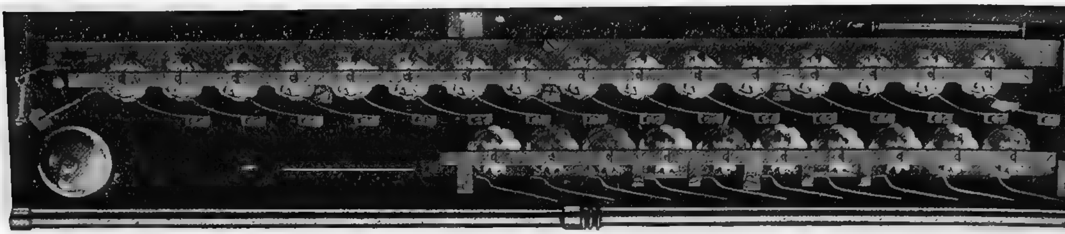


FIG. 3.

setting mechanism may be brought into gear with all the figure discs of the counting mechanism.

In the above-mentioned hinged and sliding plate there is also, in models B and C of the "Archimedes," above the row of indicator holes of the product-register, a second row of holes to register the number of turns, called also the quotient-register. On account of difficulties of construction, this mechanism has in almost all Thomas machines no carrying arrangement. But in the "Archimedes" this difficulty has been solved. The advantage of the solution is extremely important, especially in contracted methods of calculation.

(2) **Colt's Calculator.** Abridged from the German of PAUL VAN GÜLPEN

The Teetzmann calculating machine "Colt's Calculator" is a new type of the old Odhner calculating machine. The characteristic features of all

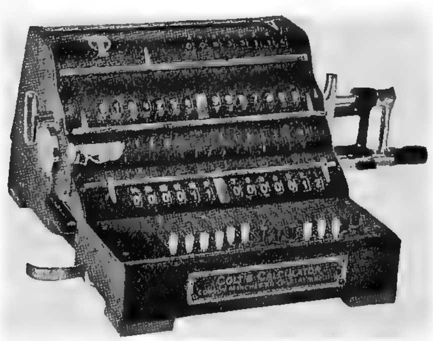


FIG. 4.

machines built on the Odhner system are toothed wheels with a variable number of teeth, in contrast to the Thomas system, which employs stepped cylinders or reckoners. The disadvantage resulting from this arrangement of the Thomas machine, namely, that the individual digits of large numbers

are, as a result of the size of the cylinders, separated from one another, and therefore difficult to read, was successfully avoided by the thin, close-set parallel discs of the Odhner system. The teeth of these discs gear with narrow toothed wheels which carry figures on their rims, so that the numbers, standing close together as if printed, are shown clearly to the eye of the operator.

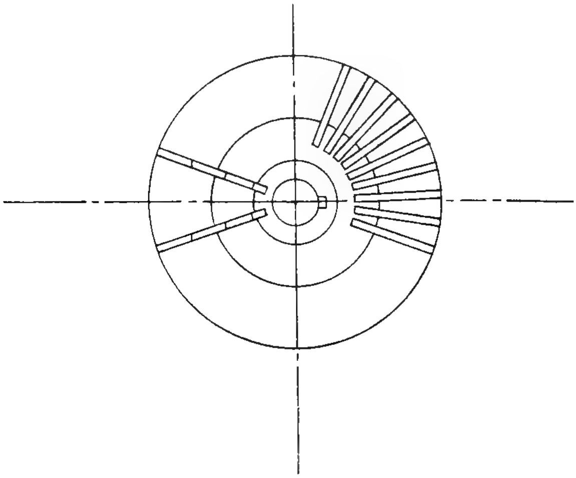


FIG. 5.—Metal Disc.

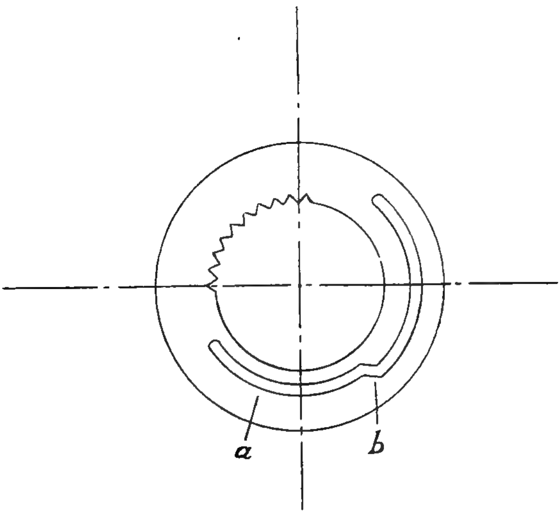


FIG. 6.—Covering Disc.

The Odhner toothed wheel consists of two parts, a metal disc with slots and a thin covering disc with a raised centre, attached so as to turn on the other (figs. 5 and 6).

In the slots of the metal disc lie steel "fingers" with a projecting catch—the movable teeth of the toothed disc.

The catches of these fingers project into the slot (*a*) in the covering disc, and follow the slot when the disc is turned. In so doing the catch follows the

crossing (*b*), and so has its distance from the centre increased or decreased. As a result of this the top part of the "finger" projects from the rim of the disc as a tooth, or conversely is withdrawn. It is obvious that by a corre-

FIG. 7.—Finger.

sponding turning of the covering disc the number of teeth on the disc may be altered from 0 to 9.

A further advantage of the Odhner arrangement was, that positive operations could be carried out by turning the handle to the right and negative ones by turning it to the left, an arrangement which seems natural, while in the Thomas system the moving of a separate lever from addition to subtraction and *vice versa* has to be carried out every time.

These advantages of the Odhner system caused many manufacturers, after the expiry of the patent, to develop the system further, and there are various machines of this type on the market.

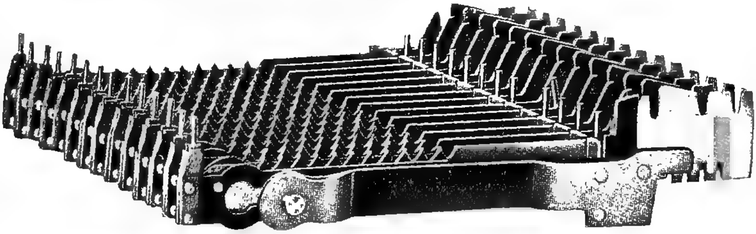


FIG. 8.—The Setting Mechanism.

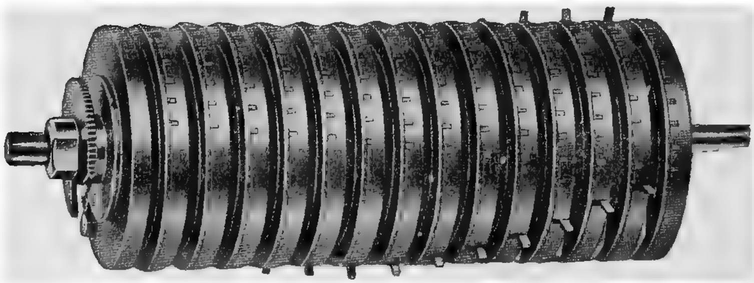


FIG. 9.—Counting Mechanism.

In all of them, however, there persists this defect, that in order to set the number of teeth on the toothed disc, the covering disc must be turned directly. In doing so the hand setting the figures must be continually raised, as a result of which the arm tires, and the number set, which must be glanced over rapidly to test the accuracy of the setting, is frequently covered.

The Teetzmann calculating machine "Colt's Calculator" makes use of a sliding bar to set the teeth, the contrivance which had worked so well in the Thomas mechanism. Hence resulted a material advantage in the manufacture of the machine, its division into the three following groups, independent of one another :—

The setting mechanism (fig. 8).

The counting mechanism (fig. 9).

The sliding carriage (fig. 10).

The setting mechanism consists of fourteen long sliding bars, which are pivoted on an axis situated in the front part of the machine. If these bars are set in position, the slots in the spade-shaped end engage with corresponding small catches in the covering discs. By pulling the bar backwards and forwards the covering discs are turned, and in this way the desired number of teeth is caused to project, corresponding to the amount of the forward push. A special toothed gearing on the sliding bar engages simultaneously with gear wheels which are fitted with digits, thus registering the number of teeth set on the disc, and likewise the number set in the calculating mechanism.

The calculating mechanism is thus coupled with the setting mechanism during the operation of setting. In order to count, the former mechanism must of course be set free again. This putting out of gear of the setting

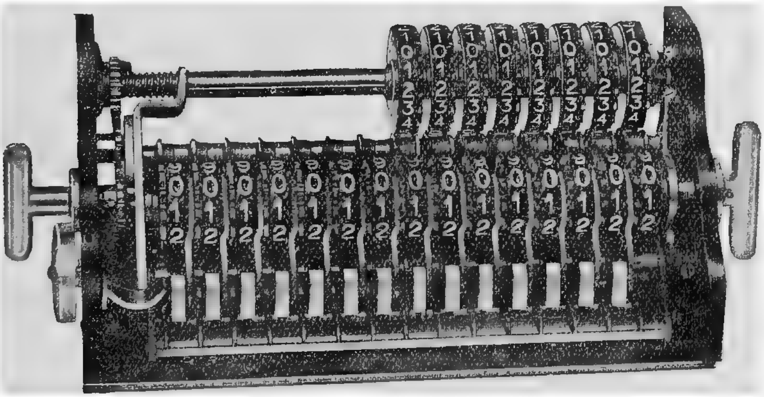


Fig. 10.—Sliding Carriage.

mechanism is accomplished automatically in the pulling forward of the driving handle. The counting itself is carried out by causing the toothed discs to revolve. In each complete revolution of these discs the projecting teeth engage with the wheels of the sliding carriage, situated opposite, and fitted with digits on their rims, and turn these wheels as many steps further on as there are movable teeth projecting. The number which appears finally indicates the result.

So far the problem of mechanical calculation appears extremely simple, nor do any difficulties appear so long as the result remains under 10; these difficulties first make their appearance in the carrying.

Supposing that the *figure* disc on the extreme right of the sliding carriage stands with the 6 in front, and that the corresponding *toothed* disc has four teeth projecting, then a revolution of the *toothed* disc in a positive direction would move the *figure* disc four figures further on, and accordingly after the 9 the figure 0 would appear. In order to obtain the correct result 10, the next figure wheel on the left must also be influenced, *i.e.* be moved on one step. This purpose is served by the carrying arrangements, on the faultless operation of which the accurate working of the machine depends.

While in all other machines of the Odhner type the most important

of these contrivances, the so-called carrying lever, is in the form of a hammer, in the case of the Teetzmann calculating machine it takes the shape of a bar sliding horizontally on two rollers. In the figure of the sliding carriage this bar can be seen clearly beneath the figure discs. As soon as the figure wheel is so moved that the 9 changes to 0, or *vice versa*, this carrying bar is pushed forward by a bent lever. The wedge-shaped point of the carrying bar presses in this position a movable pin or " finger " (the carrying pin) into the plane of the teeth of the next toothed disc, and thus causes the next figure wheel to be turned a step forward or backwards.

With the introduction of this sliding bar Teetzmann & Co. appear to have solved successfully the most difficult problem of the calculating machines of the Odhner type.

The method of setting the figure wheels at zero, which operation is necessary before beginning each new calculation, has been altered little in principle from that invented originally by Odhner. It consists in arranging the shafts so as to be movable with respect to the figure wheels, of which they form the axles. If the shaft is slid sideways a little and at the same time turned through 360°, by turning a key, small pins on the shaft catch on corresponding pins on the wheel and carry round the figure wheel, until the 0 appears again in front. In this " clearing " operation the releasing of the brake-springs, situated beside the toothed gearing of the figure wheel for the purpose of preventing " skipping " while counting, causes a clicking noise. Also, these springs oppose a certain resistance to the turning of this shaft. In " Colt's Calculator " all the brake-springs are raised at the beginning of the clearing operation, so that the clearing proceeds quietly and smoothly.

A description of the construction of the inner parts of the machine has now been given. Viewed from the exterior, what strikes one is the absence of the long dust-collecting slot in the upper part of the cover, which could be dispensed with on the introduction of the setting lever, and also the clear, close-set number-register. The figure wheels, which in other machines frequently consist of rubber with sunk digits filled up with composition, are formed of a metallic alloy, in which the digits stand out in bold relief from a black-enamelled background. As the whole number-register, set almost perpendicularly to the line of sight, is contained within a rectangle of 13 by 17 cm., all three rows can easily be taken in at one glance.

The back of the machine consists of transparent " cellon," a non-inflammable substitute for the highly inflammable celluloid, a change which has been made in the interest of smokers. Thus it is always possible to have a view of the interior of the machine, without first having to unscrew the cover.

The machine is constructed with great care, and the parts are interchangeable. It is dispatched in a dust-proof case, in which it is hung by strong springs to prevent damage by shock.

The manipulation of calculating machines is so widely known that an explanation would be superfluous. The longest multiplications and divisions may be effected in the shortest time almost without possibility of error. The brain is rested instead of being fatigued by the calcula-

tion, and the operator has the comforting assurance that no errors have escaped him.

Apart from the four simple rules of arithmetic for which calculation with the machine means simply increase of speed, calculations are made possible by the machine which on paper must be broken up into distinct computations.

(3) **The Brical Adding Machines.** The British Calculators, Ltd.

The Brical machine is a little instrument designed for adding £ *s. d.*, weights and measures, or decimal coinage. The simplest form of the machine consists

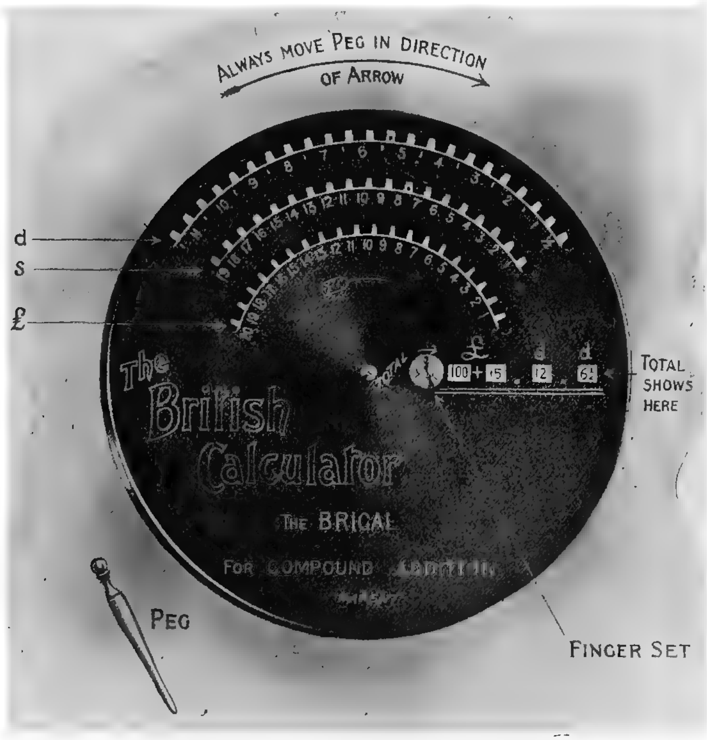


FIG. 11.

of three concentric rings, the outer circumference of each ring having a series of notches or teeth. The largest ring represents pence and halfpence, the same being printed from ½d. to 11½d. twice round the wheel, which has forty-eight teeth, each tooth representing ½d. The next sized wheel or ring is for shillings, each tooth representing a shilling, and the third wheel is for pounds, each tooth representing a pound. The wheels have no common axis, but are mounted on small bearing studs, and a slotted lid covers the whole. The slots in the lid are so arranged that the outer wheel shows up to 11½d., the shillings wheel up to 19s., and the pounds wheel up to £25. There are three squares just large enough to show one figure on each wheel, and the

total added is read from these slots. The lid is engraved under each slot for £ *s. d.*, the figures coinciding with the spaces on the wheels. Presuming the outer wheel is moved by a peg for a space of four teeth, this would show 2d. in the before-mentioned square: the shillings and the pounds wheels are operated in the same manner. When 11½d. is recorded on the pence wheel, and another ½d. added, the total shows 1s., as there is a small pin on the wheel which comes into contact with a lever having a pawl fixed to it, which engages with the teeth on the shillings wheel. The pin on the outer wheel moves the lever the space of one tooth, so that 1s. is recorded on the total. The transfer from shillings to pounds is obtained by a similar lever and pin on the shillings wheel. The wheels are independent of each other, so that pounds, shillings, and pence can be added in any order. In order to record a large amount, several wheels can be used for the pounds, one representing units, the next tens, and so on, the transfer being obtained in each case by means of a pin and lever as before mentioned.

(4) **Brunsviga Calculating Machine.** Grimme, Natalis & Co., Ltd.

On the 21st March 1912 the *Brunsviga Calculator* celebrated its twentieth year of existence, and at the same time also celebrated the completion of the 20,000th machine in the factory.

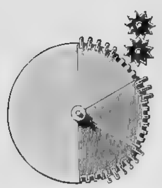


FIG. 12.—Pin Wheel of Polenus.

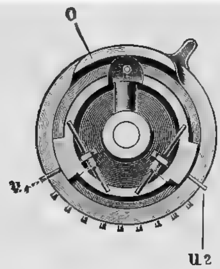


FIG. 13.—Pin Wheel of W. T. Odhner.

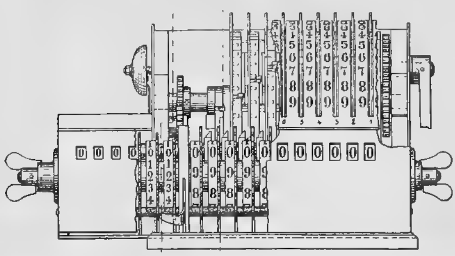


FIG. 14.—Patent Odhner of 1891.

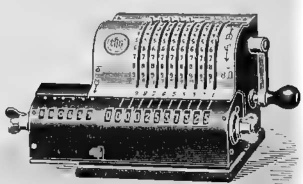


FIG. 15.—Odhner Machine.

In the second half of the last century the Russian engineer, W. T. Odhner, invented and constructed the first model of the calculating machine of the "pin wheel and cam disc" type, now universally known as the "Brunsviga."

Odhner's idea, viz. the use of pin wheels, had been described already by Polenus in his *Miscellaneis*, in 1709, and also by Leibnitz in one of his Latin treatises.

The firm of Grimme, Natalis & Co., Braunschweig, Germany, in the person of their Technical and Managing Director, Mr F. Trinks, recognised the importance of Odhner's invention and acquired it on the 21st March 1892.

Odhner constructed his machine according to his German patent of 1891.

As is usual with such early constructions, the original model still showed many deficiencies, but Mr Trinks succeeded, by numerous inventions and improvements, in raising the Brunsviga to its present level of technical perfection.

The development of the Brunsviga Calculator is best illustrated by the fact that since 1892, when first its manufacture was taken up, the firm of Grimme, Natalis & Co. have registered :

130 German patents.
300 patents in other countries.
220 German registered designs.

Most of these are Mr Trinks' own inventions, and for this reason the machine is to-day named " Trinks-Brunsviga Calculator."

The principle on which all Brunsviga machines are constructed is as follows :—

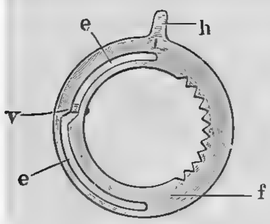


FIG. 16.

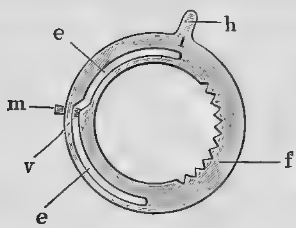


FIG. 17.

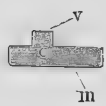


FIG. 18.

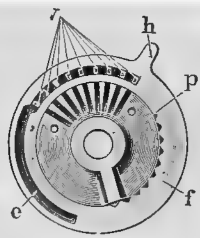


FIG. 19.

The pin wheels shown in fig. 13, whose adjustable pins *m* (figs. 17 and 18) are set by the lever *h*, are mounted on a common shaft worked by a crank. There are nine pins which can be made to project from the pin wheel as required, and when the crank is turned to rotate the shaft, these pins gear with small toothed wheels i^1, i^2 (figs. 20 and 21), which in turn gear with the number wheels E.

These number wheels E (figs. 20 and 21) carry the figures 0–9 on their periphery, and are placed on a common spindle parallel to the pin-wheel shaft.

The setting of the pins *m* (figs. 17 and 18) is produced by actuating the handle *h* of the revolving disc *f* (fig. 19), which causes the shoulders *v* (figs. 16, 18, and 19) of the pins *m* (figs. 17 and 18) to be moved into the curved groove *e* (fig. 19).

For instance, to set three pins by means of the lever *h*, pull the lever *h* until three pins project from the pin wheel, and by revolving the crank once the number wheel E of the product register is moved three places, thus the product register which previously showed an 0 now shows a 3. By turning the crank three times the sum $3+3+3$ or 3×3 is carried out and the numbering wheel registers the product 9.

In case the product consists of several digits, as in 3×4, the tens carrying device comes into operation.

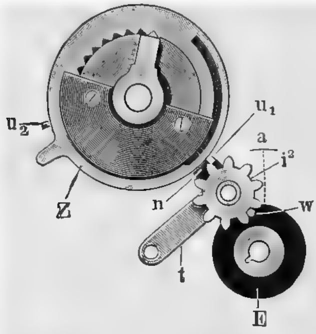


Fig. 20.

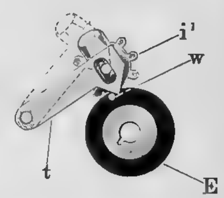


Fig. 21.

The pin w of the number wheel E displaces the hammer-shaped lever t (fig. 21) in such a way that the laterally movable pin u_1 (fig. 20) on the pin wheel Z engages with the next toothed wheel i^2 and moves this one tooth forward.

The product register is mounted on a longitudinally movable slide or carriage, arranged in front of the machine, which permits the carrying out of sums of multiplication and division in a manner corresponding to calculating with the pen on paper.

The revolutions of the crank are registered by another set of number wheels, which can also be fitted with the tens carrying device. The second counter registers in case of multiplications the multiplier, and in divisions the quotient.

Another important mechanical part is the zeroising of the registers, or, in other words, the device which brings the number wheels E back to zero. Having carried out a calculation, it is necessary, before starting a new calculation, to set the registers to "o," viz. the number wheels in the product register and in the multiplier or quotient register must be zeroised. This zeroising mechanism is illustrated in fig. 22.

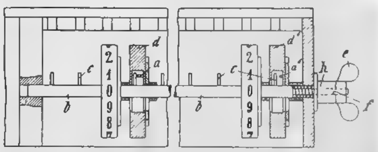


Fig. 22.

The shaft b of the counting register carries small pins c which rotate with this shaft. The butterfly nut e which is fixed to the shaft b is provided with a slant f; this slant f corresponds with a similar slant on the shoulder h. When turning the butterfly nut e its slanting side f glides on the corresponding slant of the shoulder h up to the flat top of the shoulder, which causes the shaft to be moved laterally to the right side.

The pins c moving with the shaft come into gear with the number-wheels d, d^1, which are loosely arranged on the shaft and engage pins a, a^1 carried

by these number wheels. As soon as the pins *c* of the shaft engage the pins a, a^1 of the number wheels, the latter rotate on the shaft until the butterfly nut *e* (having completed one full revolution) drops back into its original position.

By this movement of the butterfly nut *e* the shaft also slides laterally back to its normal position, and at the same time the number wheels register "0." The number wheels, which are arranged loosely on their shaft, are kept in their respective positions by means of anchor-shaped pawls and springs.

In order to remove the friction of the pawls on the number wheels and to eliminate the noise caused by zeroising, Mr Trinks has invented a device

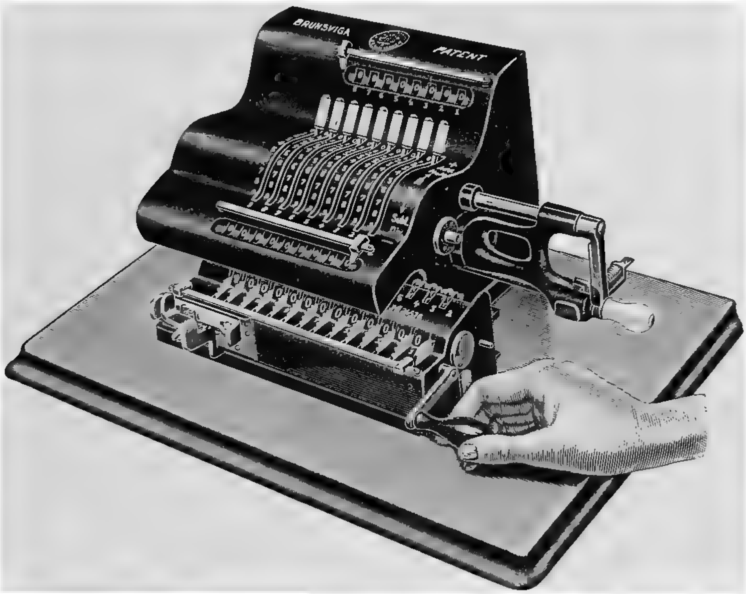


Fig. 23.—Improved Noiseless Zeroiser.

which disengages the pawls from the number wheels when the latter are being zeroised. The pawls are thrown out of gear by this device and the number wheels are brought to zero by means of toothed segments (fig. 23).

The zeroising crank is fixed on the right-hand side of the carriage, and the zeroising is effected by a half revolution of this crank. The machine is further perfected by ingenious locking devices which exclude incorrect results caused by faulty handling. The crank cannot be turned unless the carriage is in its correct position, and the carriage cannot be moved laterally when the crank is out of its normal position. Further, a reversing lock prevents the reversing of the crank (once a revolution has been commenced) until a complete revolution has been performed.

The machines with the long setting levers (fig. 24) are fitted with a similar locking device which locks the setting levers whilst the handle is being revolved.

The year 1907 brought a notable improvement of the machine with the invention of the above-mentioned long setting levers, a patent of Mr Trinks

(fig. 24). This arrangement not only facilitates the handling of the Brunsviga, but also enables the operator to have the calculation always in view for control.

Fig. 25 gives an illustration of the whole of the mechanism of the Brunsviga model J with the cover plates removed. The value set by means of the setting mechanism is made visible in a special register or indicator D. This is shown in a straight line of figures and serves as a perfect control to the operator.

The setting levers can be put back to zero singly or simultaneously by means of the crank E on the left side of the machine.

The multiplier register C is zeroised by the butterfly nut F, and the product register B in the carriage is zeroised by the butterfly nut G.

A new type, the miniature machine, Brunsvigula, was created in 1909, which does away with the noise associated with the working of the old patterns,

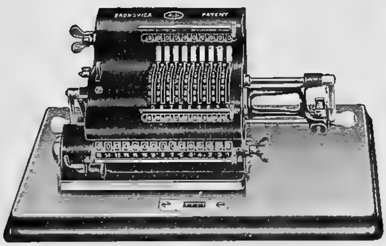


Fig. 24.

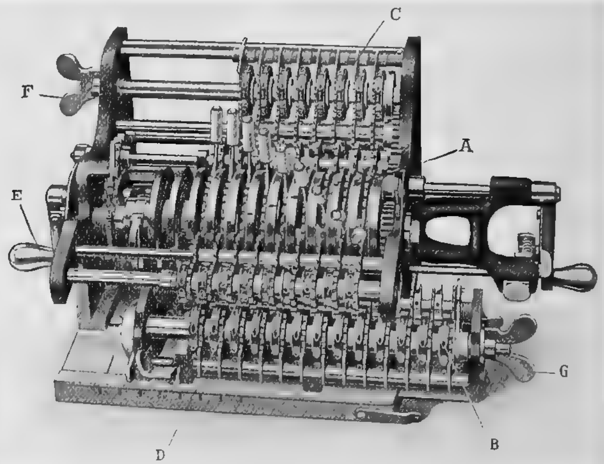


Fig. 25.

and thus renders the machine more handy to the operator. The machine is about one-half the size of the former type of the same capacity, and its construction necessitates the employment of highly trained mechanics, as the working parts are very small and must be manufactured with extreme accuracy.

The Trinks-Arithmotype was invented in 1908, as the first printing calculator for the four rules of arithmetic. This machine prints the factors as well as the product (fig. 26).

The principle of the printing mechanism in the Arithmotype is illustrated in fig. 27. The long setting lever h is connected with the segment z_1, which gears by means of a small pinion with the disc T, and which, therefore, moves the disc T by as many units as the setting lever is being moved.

The disc T on the shaft A carries on its left periphery types T^1 with the figures 0 to 9. The actuating of the setting lever sets these types to the respective figure which appears in front of the ribbon, and types the sum on the paper roll W when this is being moved in the direction of the arrow.

The contact of the paper roll with the types is effected automatically by

each revolution of the crank of the machine, which at the same time advances the paper roll from line to line.

A patented device is utilised to transfer the product from the product register to the setting levers, which makes it possible to print the product in addition to its factors.

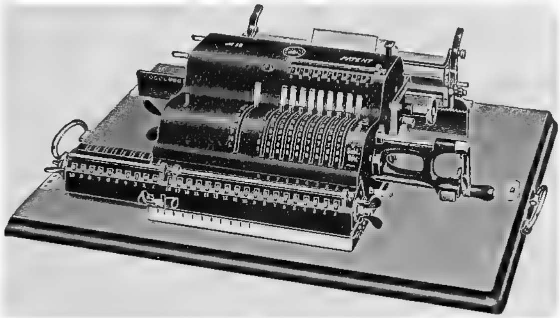


FIG. 26.

A special lever is fitted on the side of the setting levers which prints with each single factor the signs +, −, ×, ÷, £, lbs., etc., as the case may be.

A further new type of the Brunsviga is the Trinks-Triplex (fig. 28), which, as is implied by its name, is really three machines in one. It may be used either as one machine with twenty-digits capacity in the product, or the product register may be split and the machine used as two registers that are

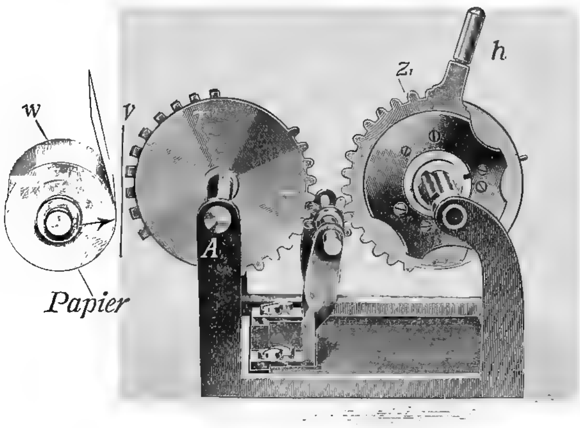


FIG. 27.

actuated by one handle. For instance, two separate multiplications can be carried out at the same time by turning the handle, and by a special device the product register can be zeroised as a whole or in separate parts.

The latest product of the factory is the machine as illustrated in fig. 29.

This is a Brunsviga miniature type with long setting levers, with the product register arranged above the setting levers and with the product counter fitted with a tens carrying mechanism.

This model claims to be the most perfect machine of the Odhner system hitherto constructed.

The multiplier register carries both white and red figures on the number wheels; white figures are registered when the machine is adding or multiplying, and red figures are registered when the machine is subtracting or dividing.

A slide provided with show-holes is operated automatically by the crank in order to display either the white or red figures of the register. This is performed without any special gearing by the hand of the operator. This automatic device affords a perfect check to the operator.

The tens carrying mechanism of the Brunsviga, also of the Brunsvigula, extends now up to twenty digits, whereas the Odhner machines only carried to ten figures.

Another interesting invention is the Automatic Carriage, which performs the shifting of the slide or carriage from one digit to another in either direction

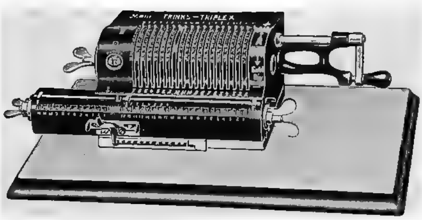


FIG. 28.

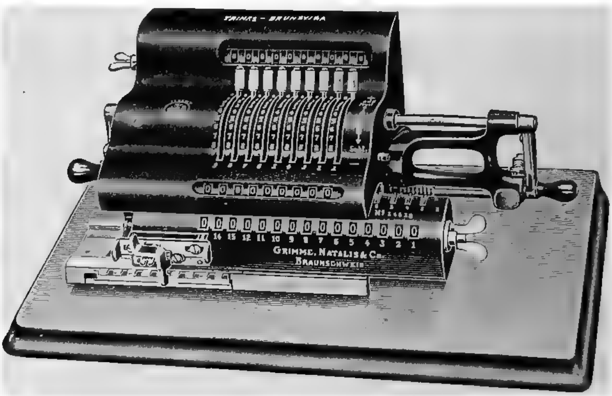


FIG. 29.

by means of a single pressure of the finger. This Automatic Carriage improvement is of great advantage, since it ensures the carriage being moved into the position desired, without necessitating the movement being watched by the operator.

The calculating principle of the Brunsviga differs from that of most other machines in so far that it follows in a natural manner the ordinary course of calculating by effecting plus and minus calculations without any change of gear.

The increasing values, viz., the results of addition and multiplication, are produced by revolving the handle in the forward (plus) direction, and the diminishing results, or the products of subtraction and division, are produced by revolving the handle in the reversed (minus) direction.

The Brunsviga Calculating Machine was first introduced into Great Britain twenty years ago at the Oxford meeting of the British Association, at which the late Marquis of Salisbury presided.

After a most careful inspection of the machine the Marquis expressed himself as being much impressed with the ingenuity of the inventor and the probable great usefulness of the machine.

The machine was one of the earliest manufactured, its number being 123, and by the courtesy of the owner (who has had the machine in daily

use ever since), this machine will be exhibited at the Napier Tercentenary Celebration.

(5) The Burroughs Adding and Listing Machine.

Reprinted from *Engineering*, May 3rd, 1907.

On this and the following pages we give illustrations of an extremely efficient adding machine, which is very extensively used in banks and clearing-houses both in this country and abroad. The machine is of American origin, but is manufactured at Nottingham by the Burroughs Adding-

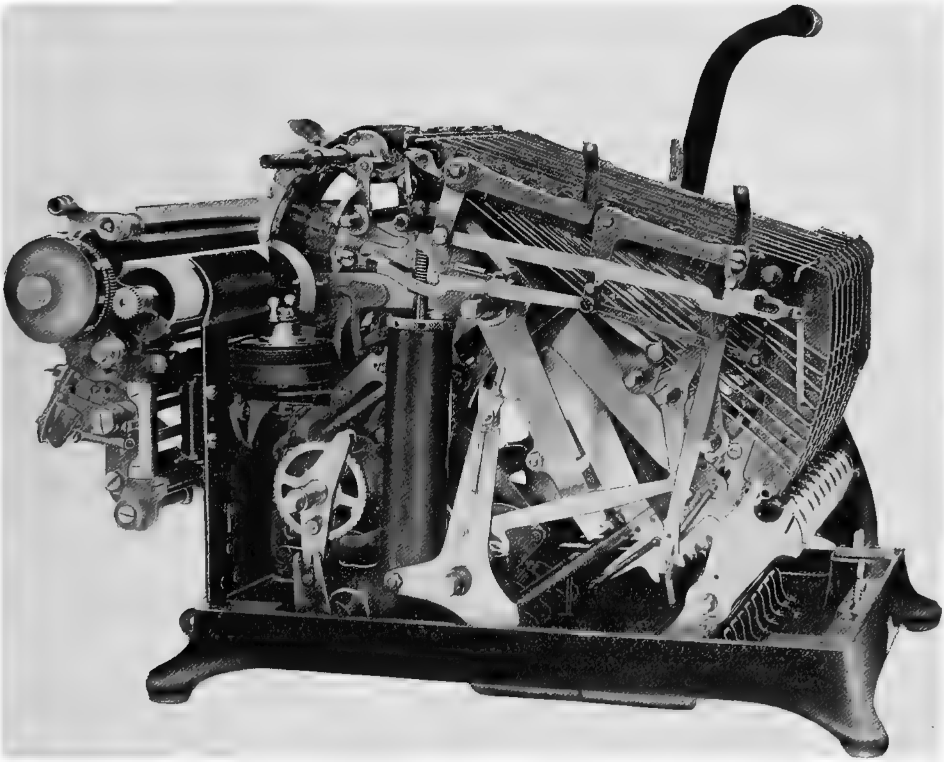


FIG. 30.

Machine, Limited, from whose works the whole of the large Continental demand is met, as well as the needs of the British market. The machine is intended to print down a column of figures, such as £ *s. d.*, and then almost automatically to print at the bottom of this column the sum total, thus relieving the clerk of all the labour of addition. In principle the machine is quite simple, the apparent complication visible in fig. 30 being due, in the first place, to the repetition of similar parts, inseparable from a machine of this kind; and, secondly, to the provision of various details, designed to make impossible the improper working of the machine by a careless or indifferent operator.

Each essential element of the machine consists of lever A (fig. 31), pivoted near the middle, carrying at the one end a set of figures from 0 to 9, held in slides by springs, whilst the other end is attached to a segmental rack B,

with which a number-wheel C can be thrown in or out of gear. The upper end of this rack is arranged to move between a couple of guide-plates D. It will be seen that a curved slot is cut in these guide-plates which is concentric with the point of oscillation of the lever A. Into this slot fits a projection from the top of the rack B, and as the other end of this rack is secured to the lever A, any possible motion up and down between its guide-plates is a true circular motion about the pivot of A. A number of slots are, it will be seen, cut in the right-hand edge of the guide-plates D, and in these slots lie the ends of a number of wires, as shown. If a key is de-

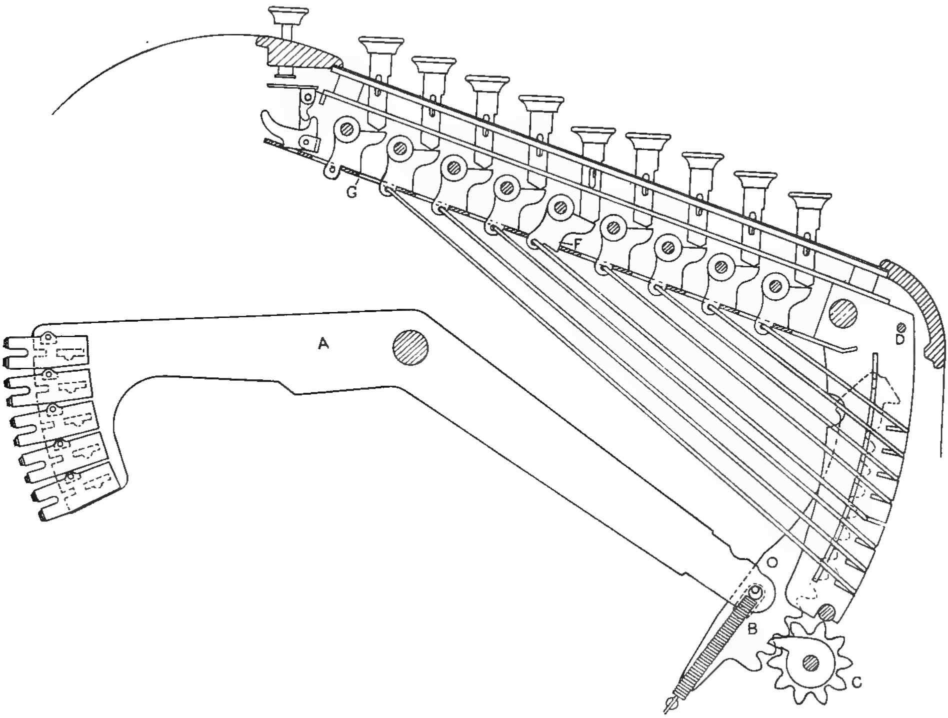


FIG. 31.

pressed, the corresponding wire moves to the left, and its bent-in end is pulled to the bottom of its slot, in which position it catches the projection shown at the top of the sector B, and thus limits its possible downward movement. With the rack thus arrested the other end of the lever A is raised, so that, of the different figures it carries, that corresponding to the key depressed on the keyboard is in position for printing. This printing is effected by the release of a small spring-actuated hammer, which, striking the right-hand end of the type-block, which, as already stated, slides in a slot in A, and is normally held back by a spring, drives it forward against the type-ribbon and paper.

The same effort which produces the downward movement of the rack throws out of gear with it the number-wheel C, which therefore undergoes no rotation during this downward motion. After the operation of printing is effected, however, the rack is raised again to its topmost position;

but prior to being permitted to take this upward movement, the wheel C is thrown into gear with it, and hence, by the time the rack is restored to its original position, this wheel will have been turned through a number of teeth equal to the number of the key originally depressed. If the series of operations just described is repeated, a second figure will be printed on the paper, and the number-wheel fed forward an additional number of teeth. Hence, if a set of these wheels is arranged in series, with suitable provision for "carrying" from one wheel to the next, as in an ordinary engine-counter, the wheels will show at any time the total of all the figures successively printed on the paper; and by suitable means this total can, moreover, be printed on the paper below the column of separate items.

This latter operation is effected by depressing the totallising key, shown at the far side of the keyboard in fig. 30, which is arranged so that no other key on the board can be depressed at the same time. The effect of the depressing of this key is to prevent the number-wheels C being thrown out of gear before the downward motion of the racks. These wheels are fitted with pawls, which prevent them being rotated backwards beyond the zero position. Thus, if in the totallising movement a wheel indicated 5, the rack in its descent would turn it back through five teeth, and would then be unable to descend further, just as if in the case previously described the wire corresponding to the number 5 key had been moved back in its slot. Hence the type end of the lever A will be in position to print the number 5, which was that on the counter. At the same time it will be seen that this counter-wheel C has been moved back to its zero position, and if moved out of gear before the racks are raised again, will read zero at the completion of the operation. Thus the taking of a total clears the machine, setting all the number-wheels to zero.

Whilst the essential principles of the machine are as just described, many safeguards are necessary to ensure its proper working. The latter involves on the part of the attendant two distinct operations. In the first place, the amount to be recorded is "set" by depressing a key on the keyboard. By pulling back the handle shown to the side of the machine in fig. 30, this sum is then printed on the paper at the back of the machine, and on the return stroke of this handle the number on the keyboard is transferred to the number-wheels, as just explained, and at the same time the keys depressed in setting the keyboard are released and return to their normal positions.

The depression of a key has three distinct results. In the first place, it moves the corresponding stop-wire to the back of its slot, as already explained. Secondly, it locks every other key in the same column; and, thirdly, it withdraws a catch which would otherwise prevent the descent of its corresponding sector B.

The locking of every other key in the same column is effected by the device shown in fig. 31. The tail of each, it will be seen, rests on the horizontal arm of a small bell-crank, the other end of which is connected to the stop-wire. As the key is depressed, the vertical leg of the bell-crank moves to the left, and carries with it a sliding-plate G, through a slot in which the lower arm of the bell-crank passes, as indicated at F (fig. 31). In

the position shown, key No. 5 being depressed, the sliding-plate G, moving to the left, has brought solid metal under the noses of each of the other bell-cranks; so that, as will be seen, it is impossible to depress any other key till the plate has been restored to its original position. This sliding-plate is constantly impelled to the right by a spring, and would fly back when the pressure on the key was removed, were it not locked by a pawl at its left-hand end. After an item has been printed, the final motion of the machine lifts this pawl, letting the plate slide back, in doing which it

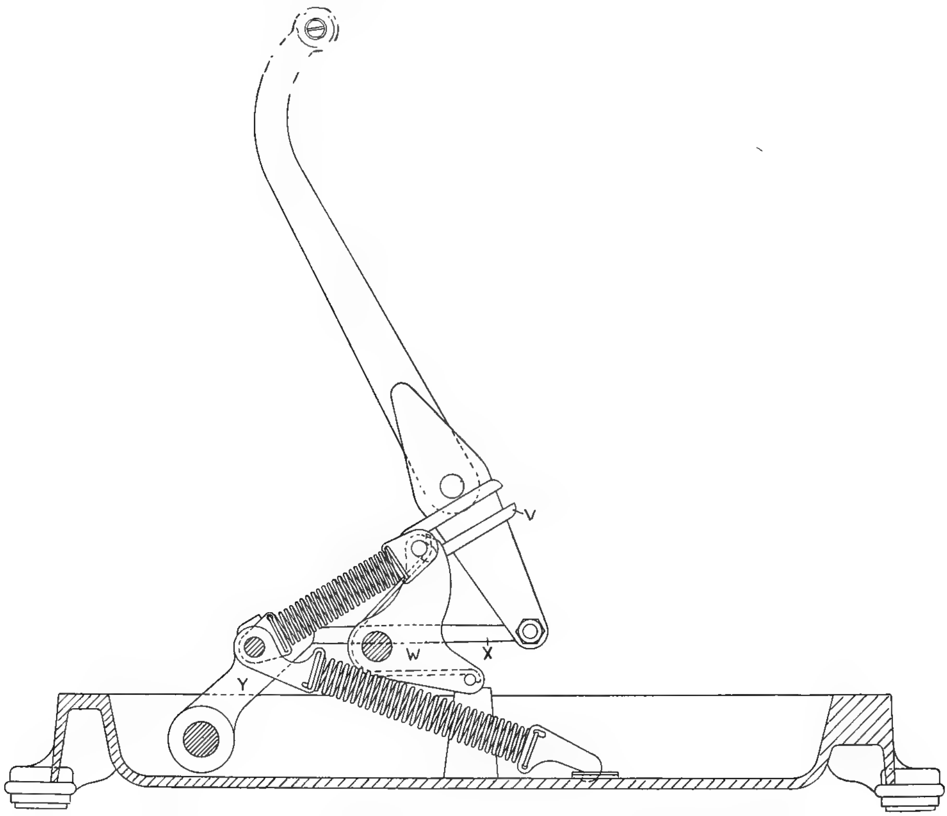


FIG. 32.

carries with it the depressed key, restoring this to its normal position. At its forward end, this plate, in being moved back by the depression of a key, carries with it, by means of a projection, the stop which, as already stated, would otherwise prevent the downward motion of the sector.

This stop, when a figure has been set, is prevented from flying back by a pawl, and this pawl is released, bringing the stop into its normal position simultaneously with the release of the sliding-plate at the end of an operation of the machine. In certain cases it is convenient to be able to repeat a number several times in succession, without resetting it. This is effected by depressing the special key, shown to the right of the keyboard in fig. 30. The depression of this key prevents the pawls which hold the sliding-plate G, on the depression of a key, from being raised at the end of an operation

of the machine, and consequently any depressed keys remain down. Provision of this nature is possible, since but very few of the various motions of the machine are positive in character, but are effected through the medium of springs. Summing up, it will be seen that the depression of a key has but three simple results. All further operations are effected by pulling back to the limit of its travel the side handle shown in fig. 30, and letting it return of its own accord. The effect of pulling over this handle is to throw into tension a series of powerful springs in the base of the instruments; these springs acting then as driving power to the main shaft of the machine. The rate at which they succeed in effecting the different operations is governed by an oil dashpot, and hence sufficient time is ensured for all the successive operations of printing and totallising to be effected in due order. It is therefore impossible for a careless operator to damage the machine by seeing how fast he can "buzz it round." The force operating the machine is quite independent of that which he exerts on the handle, and cannot exceed the tension of the springs. A notched plate is, however, attached to the handle-spindle, and, moving with it, ensures by engagement with pawls that the handle shall be pulled over to the limit of its travel every time, before being allowed to return. The handle, though it does no direct driving of the mechanism, does govern some of the movements made, since the possible motion of the spring-actuated driving-shaft cannot exceed that allowed by the motion of the handle, and the latter must therefore be carried to the end of its travel before the spring-driven shaft can effect its full travel. Moreover, if this handle is out of its normal position, it throws up a bar extending right across the machine, which locks all the keys, and prevents any being depressed until the handle is restored to its position of rest.

Referring to fig. 32, it will be seen that the handle, by means of the link X, pushes over the lever Y. This lever is pulled towards the front of the machine by four strong springs hooked into the bottom plate, as indicated, and, by a set of springs, such as Z, pulls over, in its turn, the bell-crank W. It is this crank which really actuates almost the whole of the mechanism of the machine. It is coupled to Y by springs, as already stated, and moves to the left under the influence of these only. Its return stroke to the right is, however, made under the thrust of the fork V, which is pivoted to Y. Hence the driving power of the machine on its return stroke is provided by the springs connecting the lever Y with the base of the machine, and in the forward stroke by the springs between Y and W. On both strokes, therefore, the machine is spring-driven. A dashpot, not shown in this figure, but clearly visible in fig. 30, which represents the machine partially dismantled, controls the speed of the machine on both strokes.

We have already explained that in the operation of listing a series of items which are ultimately to be added up, the first action of the machine is, through suitable linkwork, to shift all the number-wheels clear of the descending racks. To this end the whole set are mounted on a frame extending right across the machine. This frame is itself mounted on pivots, so that it can be swung in or out from the racks. As soon as the handle has been moved over to the full extent of its travel it is automatically locked here, and prevented from returning until the operation of printing has been

effected. On the return stroke of the machine the wheels are swung into gear with the racks, which, in ascending, turn these wheels round through a number of teeth equal to the number of notches, past which the rack has been allowed to fall till brought up by the stop-wire. In order that these wheels shall always show the total sum registered by the machine, a "carrying device" is necessary from the wheel corresponding to the units place, to the tens place, and so on. This carrying device consists, in the first place, of a cam or long tooth—keyed to the number-wheel C, fig. 31. This cam does not, as in an engine-counter, rotate directly the wheel next above it,

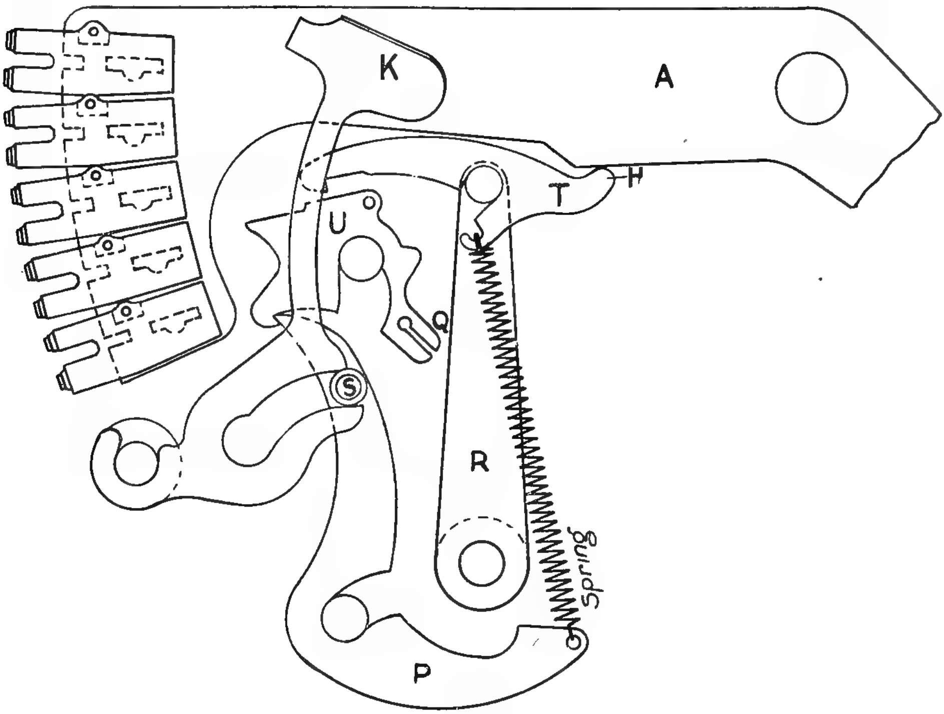


FIG. 33.

but merely releases a stop, which, when no total is being carried, limits the rise of the succeeding rack. Hence, if a "carrying" operation is to be made from the units to the tens wheel, the cam on the former displaces a stop in the path of the tens rack, and, as a consequence, on the return stroke of the machine, the tens rack rises beyond its normal position to a height equivalent to the pitch of its teeth. While the racks are rising (during the operation of listing) the number-wheels, as already stated, are in gear with the racks; hence, in the above case, the tens wheel rotates one tooth more than it otherwise would have done.

In the operation of totalling, it will be remembered that the relation of the number-wheels to the racks is reversed; that is to say, they remain in gear during the down stroke of the racks, and are thrown out of gear on the return. As the racks in totalling fall to a distance limited by the wheels rotating backwards to the zero position, it is essential that these racks shall

be in normal position before a total is effected, and hence provision is made by which, if any rack is in the high position due to its having "carried over" from one wheel to the next, a stop is thrown into action which makes it impossible to depress the totalising key at the left hand of the machine. By making an idle stroke of the machine the racks are restored to the normal position, and a total can be taken. This idle stroke of the machine, moreover, feeds forward the paper on which the items are listed, so that a space intervenes between the list of items and the total printed by the next movement of the handle. This space serves the useful purpose of distinguishing a total from one of the individual items, the column of items being always separated from the total by this space.

We have said that in "carrying over," the rack which effects the operation rises one tooth beyond its normal position. This is possible, because, as will be seen from fig. 31, the rack is connected to the swinging beam A by a pin working in a slot. A spring tends to throw the rack up and bring the pin to the bottom of the slot. When no "carrying over" is to be effected, the beam A, in moving back to its normal position, carries with it the rack B, but the latter is stopped in its upward movement by a catch before the beam A has completed its stroke. This the latter does in stretching the spring connecting it with B, and comes to rest finally with the pin at the top of the slot. If, on the other hand, the long tooth on the preceding wheel has removed the stop in the path of B, the latter moves with A till the latter has completed its stroke and comes to rest with the pin at the bottom of the slot, and, therefore, one pitch above its normal position. Each of the swinging beams A is connected on its right-hand side with a spring, pulling it downwards. A bar extending right across the machine prevents any one of the beams descending, until it has been swung out of the way by pulling the operating handle. When this bar has been swung clear, any one of the beams which may have been released by the depression of a key is pulled down by its spring till brought to rest by the stop-wire connected to the depressed key. On the return stroke of the machine, the bar, already mentioned, is swung up to its original position, carrying with it all the beams which have been displaced; and when these are home, they are locked there by a set of pawls, each of which is released only by depressing one of the corresponding keys.

The swinging beams A are bent in the horizontal plane, so that whilst their type ends are set at $\frac{1}{8}$-in. centres, their other ends are $\frac{3}{4}$ in. apart. At its type end each beam has mounted on one side of it a set of five little blocks, which move in slots, and are held back towards the pivot of the beam by springs. Each block carries two types, the five giving all digits from 0 to 9, whilst a set of little hammers, spring-actuated, lie between each set of beams, and, if released, will drive forward the block in front and print the corresponding character on the paper. The release gear for these hammers is shown diagrammatically in fig. 33. There are a series of pawls T mounted side by side on a pin, which is carried by two links swinging about a centre R. If this link is swung forward, it can, it will be seen, catch a second pawl U, provided always that the forward end of T is allowed to fall behind the catch. If the main swinging lever A, fig. 31, corresponding to T, is in

its normal position—that is to say, if no one of its corresponding keys has been depressed—the tail H of the pawl T is prevented from rising by the underside of this lever, and as a consequence its forward end cannot catch hold of U. Hence, on the return stroke of the frame on which T is mounted, U remains unaffected, and the striker P, which drives the type-hammer by the roller S, remains in place, and consequently no printing is accomplished as far as that particular element of the machine is concerned.

If, on the other hand, a key has been depressed on the board in the row corresponding to the pawl T, the sector end of the corresponding lever falls, and its type-carrying end rises, so that the tail H of the pawl T is no longer kept from rising. The main lever having been brought into position by the fall of the sector against its stop-wire, as already explained, the further operation of the machine swings forward the frame on which is mounted the pawl T, which, as its tail can now rise, grabs U, and, on its return stroke carrying this with it, releases P, which, driven forward by its spring, strikes the hammer sharply against the back of the type-block, and the corresponding character is accordingly printed. The arrangement of pawls and levers P, U, and T is repeated for each place in the pounds, shillings, and pence column, the whole set being mounted side by side. As stated above, the pawl U is, in general, never raised unless a key has been depressed in the corresponding column of the keyboard. If, however, it is desired to print the sum of £500, say, then it is convenient that the zeros shall be printed automatically, without requiring to be set on the keyboard, for which, in fact, no provision is made. To effect this the tail Q of U for the hundreds column has a projection on its right-hand side, which extends over the tail of the U pawl for the tens column. If, then, the U pawl for the hundreds column is raised by its corresponding piece T, its tail Q pushes down the tail of the U pawl for the tens column, and thus releases the corresponding striker P. Similarly, the raising of the U pawl for the tens column releases also the striker for the units column; and thus, in the case taken, the sum £500 will be printed, though only one key has been depressed on the keyboard.

(6) **The Comptometer.** Felt & Tarrant Mfg. Co.

The Comptometer was brought out about 1887 by the inventor, Mr Dorr E. Felt, Chicago, U.S.A., and is now manufactured and sold by the Felt & Tarrant Mfg. Co., Chicago.

It claims to be the first successful key-operated adding and calculating machine. Prior to its appearance some crank-operated machines had been manufactured and sold; but the practical operation of these machines was confined to calculations involving multiplication and division. It is designed to be rapid and efficient in all arithmetical operations. In calculating, the results are obtained by simply depressing the keys, without any auxiliary movements. This one motion is naturally conducive to speed, and for calculations with factors up to six by eight digits, which covers the range of the great majority of commercial problems, the Comptometer is highly satisfactory. The latest model embodies the principle found in the earliest models, *i.e.* a bank of keys actuating a series of segment levers which in turn actuate the numeral wheels of the register. A positive stop,

thrown into position by the key, determines the length of travel of the lever. On the end opposite the fulcrum of this lever is a rack tooth segment which engages a pinion carrying a ratchet, which in turn engages a pawl fastened to a gear; this gear through a train of two other gears rotates the registering or accumulator wheel in accordance with the key struck.

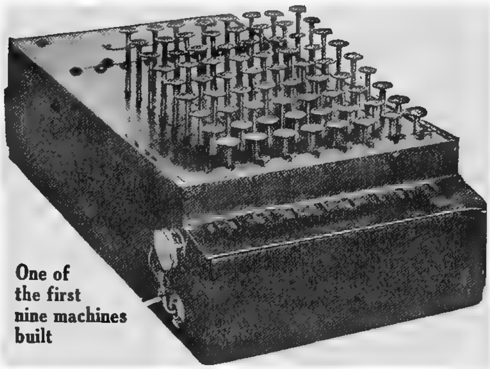


FIG. 34.—Early Type.

The carrying of tens is accomplished by power generated by the action of the keys and stored in a helical spring from which it is automatically released at the proper instant to perform the carry. To guard against over-rotation of the accumulators in either direction from the impulse of the prime movers or from that of the carrying mechanism, positive stops are also provided.

Improvements, however, have been added from time to time which, together with refinements of construction, have contributed much to the speed,

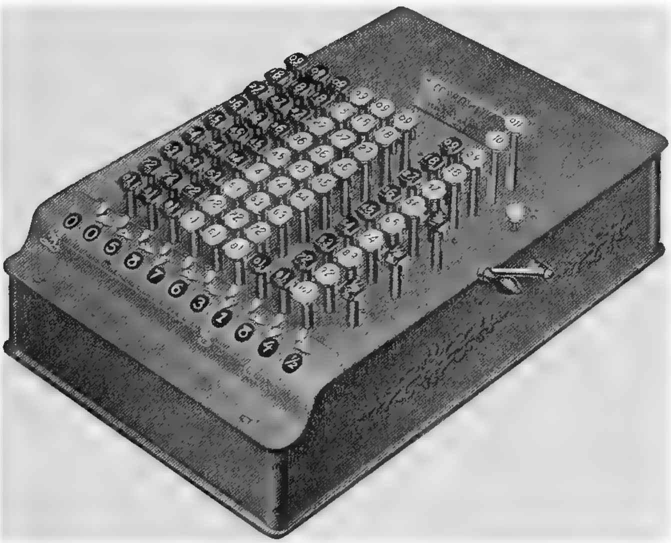


FIG. 35.—Modern Machine.

ease, and accuracy of operation in the modern machine. Notable among these improvements is the duplex feature introduced a few years ago. Prior to its invention only one key could be operated at a time. This meant that if a second key was struck before the one previously struck had returned to normal position an error might result; but with the duplex machine there is no need for the exercise of care in this respect, as it provides for the simultaneous operation of two or more keys in different columns. Besides simplifying the operation the duplex feature adds greatly to the speed and accuracy

of the Comptometer. It facilitates calculations in multiplication and division in a remarkable degree, since as many keys as can be conveniently held by the fingers of both hands may be struck at the same time. Thus in multiplying, say, 47685 by 3457 it is only necessary to strike the keys representing the latter factor five times in the unit's position, eight times in the ten's position, six times in the hundred's position, and so on across, when the answer appears in the register.

The latest improvements in the Comptometer appear in a recent model known as the Controlled-Key Comptometer. In any machine not wholly automatic there is always a human element to be taken into account—an element always prone to error. It was for the purpose of eliminating, to the last possible degree, the chance of error from this source—errors due

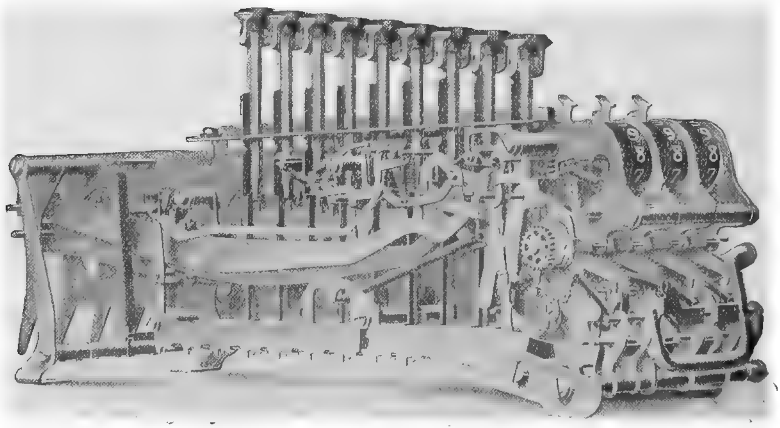


FIG 36.—The Mechanism.

to the inexperience of beginners and the carelessness of experienced operators—that the Controlled-Key was devised. This safeguard consists of :—

1. Interference guards at the side of the keytops to prevent accidental depression of a key at either side of the one being operated.

2. The automatic locking of all other columns when a key in any column is not given its full down-stroke.

3. An automatic block against starting any key down again until the up-stroke is completed.

The illustration fig. 37 shows how the Controlled-Key acts under a fumbled stroke. It will be noted that in attempting to depress the white-topped key the stroke was misdirected so that the finger overlapped on the black-topped key far enough to touch and bear down on the interference guard. The black-topped key is not affected by this contact, because the Controlled-Key is built in two parts, and pressure on the part to which the interference guard belongs does not depress it. Unless a key is touched squarely enough to first depress the keytop to a level with the interference guard it will not go down. The effect of this is that in regular operation it is practically impossible to accidentally touch two keys at once so as to put them both down with one finger on the same stroke. Thus it can be seen how completely the Controlled-Key guards the operator against the consequence of fumbling.

In order to perform the proper functions and add correctly, the keys of the Comptometer must, of course, be given their full determined travel on both the up- and down-stroke. As with the typewriter, the operator soon learns the correct stroke, which quickly becomes an automatic habit, and is able to manipulate the keys at high speed with remarkable accuracy. A beginner, however, in trying to go too fast at the start, might by a slurred or partial key-stroke make it add a wrong amount. Such faults, whether due to inexperience or carelessness, are overcome by the Controlled-Key, which, if not given its full down-stroke, causes the keys in all the other columns to lock up instantly; and when the operator goes on to the next key after such a misoperation, he finds it will not go down. On looking at the answer register he sees in one of the holes a figure standing out of alignment toward him. This indicates the column in which the fault occurred. Now, by noting the last figure added in this column, he can tell at once which key was misoperated. Correction of the error is made by simply completing the unfinished stroke of the partially depressed key, after which the release key is touched to unlock the machine.

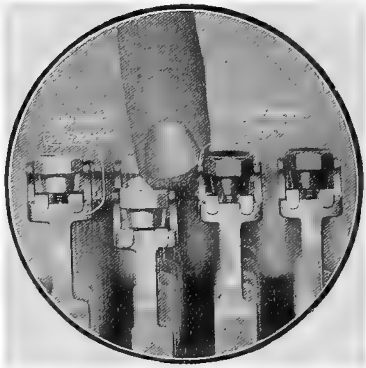


FIG. 37.—Interference Guard, and Cushioned Key-tops.

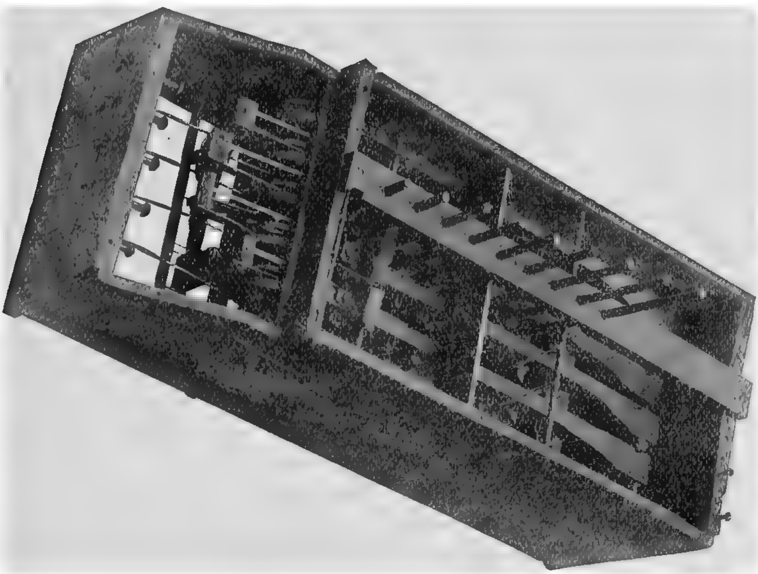


FIG. 38.—Macaroni Box.

Another safety feature of the Controlled-Key is its automatic prevention of an incomplete up-stroke. Should the operator, when striking the same key twice or more in rapid succession, attempt to start it down again before letting it clear up, he will find it impossible to do so. Once the key has started

up, it automatically locks against reversal at any point short of its full upward travel.

Briefly summarised, the effect of the Controlled-Key is to automatically prevent the operator from accidentally overlooking any errors that may arise from imperfect operation.

The tendency in invention of office appliances is steadily toward more complete automatic control of mechanical functions, and in its development the Comptometer seems to have followed this line.

(7) **Layton's Improved Arithmometer.** Manufacturers: Charles & Edwin Layton.

In the year 1883 Messrs C. & E. Layton exhibited the first arithmometer of English manufacture as the agents of Mr S. Tate, and soon afterwards acquired the patents connected therewith.

The following is extracted from a paper read at the Society of Arts, 3rd March 1886, by Professor C. V. Boys, A.R.S.M., descriptive of this machine :—

" I have said that the machine referred to is in appearance identical with the de Colmar machine. This refers to the general design and to the outside. When opened, great differences are at once apparent, the most important being the substitution of the best English for what can hardly be considered the best foreign work. It is impossible to speak too highly of the beautiful finish, the accuracy of construction, or the excellent materials which are employed in every part. So far the machine might be nothing more than the French machine better made. There are, however, improvements in detail in the design. In the first place, the erasing mechanism is, in practice, far more convenient than in the French machine. In the place of a long rack which pulls each dial round until, in consequence of an absent tooth, it stops at 0, an operation performed by twisting a milled head against a spring for one set of dials, and another in the same way for the other set, it is merely necessary to jerk a handle one way to erase one set of numbers, and the other way to erase the other set. The dials are brought accurately to zero by a long steel rod, acting on cams, exactly in the same way that the second hand of a stop-watch is set back to sixty.

" Another improvement is the removal of the stops, or cams and cam-guards, which prevent the dials and auxiliary arbors from overshooting their mark in obedience to their momentum. These guards, which act much in the same way that the Geneva stop prevents overwinding of a watch, suddenly bring the dials to rest. In place of these, a series of springs are employed, under which these parts move stiffly. This, at first sight, seems inadequate, in view of the great speed at which the machines are run. I have done my best to try and make one of these overshoot, but without success. I thought it would be interesting to find how far the dial must really move before the spring brings it to rest. I therefore made the following measures (on the C.G.S. system) :—The moment of inertia of the dial and its attachments is 10·9, and of the secondary axis and wheels 6·7. If

we take a working speed of four turns of the handle a second, we shall find that the angular velocity of these parts is in radian measure 16π or 50·4, and therefore the energy of motion is 22,370 units. The springs are adjusted until they resist a force equal to the weight of a kilogram applied to the teeth, which represents a turning moment of 784,800 units. These figures make the greatest possible amount of overshooting to be about $1\frac{1}{2}^\circ$. Now, as no error could be introduced unless an angle approaching 18° were reached, it is evident that the factor of safety is fully 10, and that any fears as to the efficiency of this break are unfounded. The break has been found an efficient means of checking the motion of heavier things than the wheels of a calculating machine.

"Against this break may be urged the fact that more mechanical work is spent in driving the machine, but this is so slight that it can hardly be urged with propriety. The remaining improvement relates to the method of holding the carrying arm in its working or its idle position. To what extent

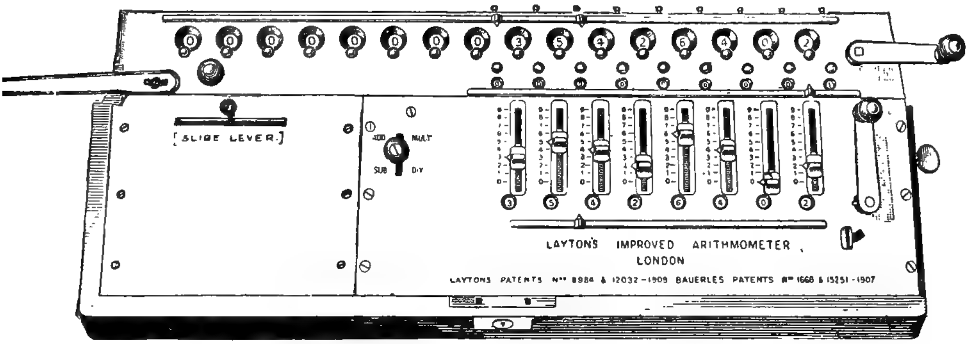


FIG. 39.

the old-fashioned double spring is likely to fail I am not in a position to say; I think I may safely say that the simple spring that takes the place of this double spring can never fail."

During recent times many other important patented improvements have been incorporated, and the instrument is now known as Layton's Improved Arithmometer.

1914 Model

Important New Features

Lightness.—Special attention is drawn to the introduction of modern alloys with small specific gravity combined with great strength, making the instrument much more convenient to use and handle. Without in any way impairing the strength, durability, or reliability of the machine, it has been found possible to produce an arithmometer of one-half the weight of the ordinary model, which is, therefore, much more convenient to carry. No alteration has been made in the size or shape of the instrument. The metal is non-rusting and not affected by acids. It is, therefore, particularly suitable for hot or wet climates. Machines constructed of this metal work with the minimum of noise and are light running.

The Markers.—Hitherto markers have been set to the figures required one by one, and have been returned to zero in like manner. The new invention allows these operations to be performed as before; but in addition a button is provided, which, on being pressed, returns all the markers to zero at once. Thus several operations are combined conveniently, and a fruitful source of error to following calculations avoided. The working parts of this device make it almost impossible for a marker to rest between two digits.

Show Holes in connection with the last invention have been added, so that the figures can be set more quickly by the markers and checked more easily.

The Slide Lever.—To move the slide in previous models of the arithmometer required two distinct movements, viz., to raise, and to propel. By means of an arrangement now invented, this double movement is performed by simply pulling a lever. The slide can be moved in either direction, and falls automatically into its correct position.

The Regulator.—Hitherto the handle has been actuated by the left hand, which is also needed for the slide. In practice this has been found to be inconvenient, particularly when the short method of multiplication is used. The new invention provides a method by which the regulator can be controlled by the right hand, as well as by the left hand as hitherto.

(8) **Hamann's "Mercedes-Euklid" Arithmometer.** By O. Sust, Kgl. Landmesser in Berlin. Translated by W. Jardine, M.A. From *Zeitschrift für Instrumentenkunde*, 1910.

Herr Ch. Hamann, of Friedenau, Berlin, is well known as the designer of the "Gauss"[1] arithmometer, whose easy manipulation has made it a favourite for certain kinds of computation. The same inventor has since designed another machine depending on the addition principle, which has now been placed on the market under the name of the "Mercedes-Euklid."[2] Its invention represents an attempt to overcome the numerous defects[3] in existing mechanical calculating systems, especially the incomplete carrying over of tens and the difficulty of division, both of which forced the user of the machines to be continually on guard, and consequently quickly tired him. In the Euklid, not only are these faults got rid of, but so many innovations and improvements have been carried out that it represents an entirely new design, differing fundamentally from those already in use. The mechanical carrying over of tens is continued right up to the highest place, so that correction of results is never necessary. Further, the quotient (or "rotation") mechanism is fitted with an arrangement for carrying over tens, which is

[1] Berlin, Kgl. Landwirtschaftliche Hochschule, June 1910. Compare the descriptions in the *Zeitschrift für Instrumentenkunde*, xxvi., S. 50, 1906; xxix., S. 372, 1909.

[2] The machine is protected by D.R.P. No. 209,817, and the notification number 35,602. It is sold by the "Mercedes" Bureau—Maschinen Ges. m. b. H., Berlin S.W. 68, Markgrafenstrasse 92/93.

[3] Compare O. Koll, *Die geodätischen Rechnungen mittels der Rechenmaschine*, Halle, 1903, Vorwort, Abschnitt 4; also the report "Neuere Rechenhülfsmittel" in *Z. f. I.*, xxx., S. 50, 1910, in which mention is made of the tables of O. Lohse and reference made to the disadvantages of detailed division with calculating machines, which disadvantages cannot be quite got rid of by the use of tables of reciprocals.

found to be especially useful in some kinds of calculation. Owing to the proportionately small size of the machine, a desirable compactness is obtained, and, at the same time, attention is paid to the convenient arrangement and easy manipulation of all levers. Provision is also made for every means of ensuring against incorrect manipulation. A special merit is the noiseless action, which permits of the use of the machine in large offices without thereby disturbing those working near. In spite of all these advantages, considerations might be raised against the introduction of a new addition arithmometer, since serviceable multiplication machines have been constructed[1] which demand, in general, less crank-turning than this one to form a product. But this disadvantage is small in comparison with its noiseless action, and with the further advantage which the Euklid possesses that an entirely automatic division of any chosen numbers may be per-

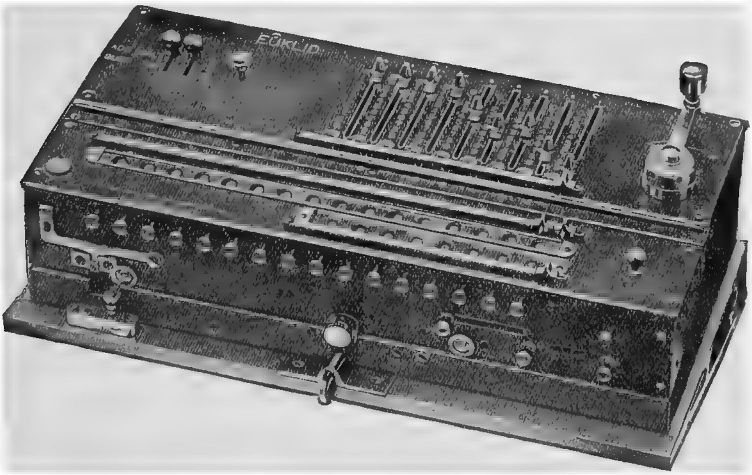


FIG. 40.—($\frac{1}{4}$ actual size.) Appearance of the machine.

formed without any attention on the part of the user of the machine. The most conspicuous defect of all systems hitherto constructed is thereby got rid of.

Fig. 40 shows the external appearance of the machine. The rectangular metal box, which is so arranged on a wedge-shaped base that the upper part is slightly tilted towards the front, is about 37 cm. long, 18 cm. broad, and 8 cm. high; it weighs 12 kg., so that the machine is easily carried about and may be set up anywhere. The upper part to the left of the crank K contains the slot mechanism, the ingenious arrangement of which made it possible to place the nine slots at intervals of only 16 mm. apart. The numbers indicated by the zigzag line of markers F are shown again in a straight line in the corresponding viewholes M. In the forepart we see the two rows of viewholes (P and Q) of the product and quotient mechanism (closed against dust by glass strips). The carriage containing this mechanism, as in all calculating machines, can be pushed for multiplication and division purposes in a longitudinal direction to positions opposite the slot mechanism. On pushing,

[1] Multiplication machine of Steiger and Egli, described in *Z. f. V.*, xxviii., S. 674, 1899. Compare also Koll, S. 20 of same.

the sliding carriage moves, without jumping or rattling, on rollers along guides in the machine frame, in such a way that the possibility of dust entering the mechanism is reduced to a minimum. Every calculation is begun with the highest place, and the carriage is pushed for this purpose to the right by means of the knob G_2 until it reaches the desired position. The succeeding motion towards the left during the calculation is self-acting. The sliding knobs G and G_1 are used for the effacement of the quotient and product. The following more detailed description will explain the manipulation and working of the pair of operating levers U and U_1, as well as of the other single parts of the machine.

The action of the slot mechanism, which rests on an entirely new principle, is explained by the diagrammatic fig. 41. Under the markers F (fig. 40) lie, parallel to each other and prevented by guides from being laterally displaced, ten racks Z_i, which are linked to a proportion lever H. The motion of a connecting rod *pl* from the crank axle is communicated to this lever, causing

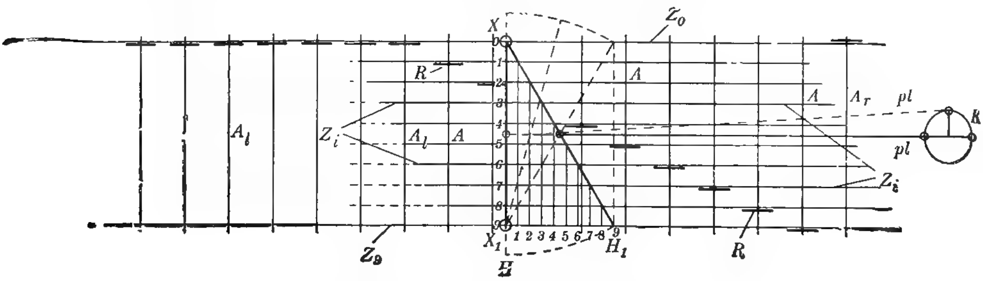


FIG. 41.—Action of the slot mechanism.

it to swing round one of its extremities, *e.g.* X, so that the racks Z_i are displaced by an amount corresponding to their distance from the pivot of the lever. In all addition processes this pivot lies on the rack Z_0; the lever then turns from H to H_1, and gives to the racks displacements corresponding to their numbering. If now, by means of the markers F (fig. 40), the ten-toothed pinion wheels R, travelling along square axles A, are placed over the corresponding racks, then they rotate by so many units in either direction. A special coupling secures that only a forward motion is communicated to the mechanism, while a reverse motion has no effect. By using the racks of the slot mechanism and dispensing with a reversing movement of the carriage, which would demand a more complex arrangement for the carrying over of tens, the slot mechanism becomes especially useful for the carrying out of subtractions. The procedure [1] previously followed in calculating with other machines of substituting for the reverse process in subtraction and division the process of setting up and adding the complements [2] of the tens is put to practical use in the simplest possible manner. By means of a reversing gear, the pivot of the lever may be placed on the rack Z_9 at the point X_1, so that this rack, which previously covered the greatest distance

[1] W. Veltmann, "Über eine vereinfachte Einrichtung der Thomasschen Rechenmaschine," *Z. f. I.*, vi., S. 134, 1886.

[2] Hr. Hamann has applied the same principle in the "Mercedes-Gauss," where the mechanical process is really less simple.

(nine units) now stands still, while Z_0 is moved through nine units. In both cases, and naturally for all intervening racks, the sum of the two motions will be nine units. A simple example will explain this process. Let the six markers F to the right be placed on the number 249,713, and the rack Z_0 be locked, then a turn of the crank will cause this value to appear on the carriage indicators P, which previously showed the value 0. To subtract the same number, we now reverse, so that the lever H rotates about X_1 on Z_9. In this way the nines complement 750,286 is added, and as result we get 999,999 instead of 000,000. The error arising in this way is got rid of by raising the units place by one. This is done by an attachment on the rack Z_0, which causes an axle A_r, situated to the right of the last of the slot axles and fitted with a rigidly attached wheel, to make a complete revolution in every subtraction, and so effects a carrying over of tens to the left, thereby raising the units place by one. Further, to the left of the nine slots, and opposite the viewholes of the carriage, lie other axles A_l, with fixed pinion wheels, which all turn when Z_0 is displaced (and therefore in all subtractions) by nine teeth, equal to nine-tenths of their circumference, whereupon nines appear opposite them in the carriage viewholes. Through the progressive carrying over of the ten these are all finally changed into nothings, and the correct result is got. The subtraction of the two equal numbers is carried out by the machine in the following manner :—

$$
\begin{array}{r}
249\ 713 \\
+\left\{\begin{array}{r} 750\ 286 \\ \ldots\ 999\ 000\ 001 \end{array}\right. \\
\hline
1 \ \ldots\ 000\ 000\ 000
\end{array}
$$

The one disappears, as it is carried over to the end part of the mechanism.

The reversing process is brought about by the lever U (figs. 40 and 43), which pushes the bolt *s* into a corresponding opening in the rack Z_0 or Z_9, while it leaves the others free. If, as in fig. 43, the rack Z_9 is locked, the pivot of the proportion lever H lies on it, and therefore subtraction results. The position of the bolt *s* can only be changed when the racks are in their initial position, as otherwise it finds a check in the opposing racks. A movement of the crank, on the other hand, can only follow if the lever U is completely shoved home, as otherwise both racks are locked by the bolt *s*. Consequently the reversing lever is converted into a safeguard against improper usage. The number cylinders in the viewholes M, which show in a straight line the numbers already indicated by the markers F, are fitted on axles W_s (fig. 42) provided with a slow worm. Against these press a pointer which is attached to the markers. A displacement of the marker F therefore causes a rotation of the axle, whose amount corresponds to the displacement, *i.e.* a change from one number to the next on the number cylinder is coincident with a displacement of the marker by a unit. In order that they may be set more easily and definitely, the markers F are provided with rollers which are pressed by a spring into grooves on the underside of the cover. The racks are set in motion by the connecting rod *pl* from the shaft W_1, which is coupled by toothed wheels to the crank shaft W. The action of the slot

mechanism is rendered more free and less liable to friction by a suitable arrangement of the proportion lever H.

Exactly opposite the slot axles lie, in the forepart of the machine, the axles a_1 of the carriage mechanism; both carry on their facing ends similar

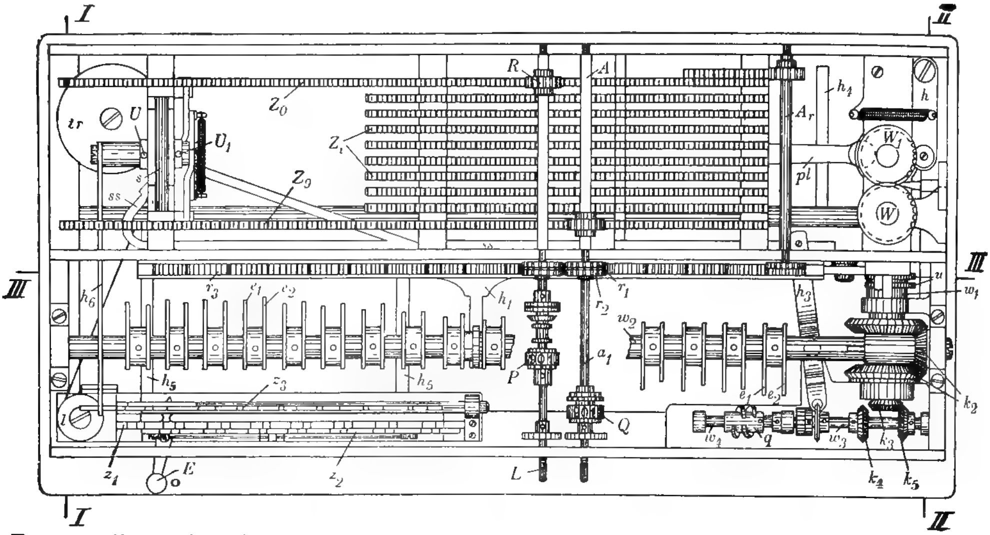


FIG. 42.—(⅓ actual size.) Appearance of the whole machine from above after removal of the cover. The proportion lever and all the slot and carriage axles except two are omitted.

ten-toothed wheels r_1 and r_2. Under these are placed on the beam b (fig. 44) broader cog wheels r_3, which can be engaged simultaneously with r_1 and r_2 and thereby rigidly connect both sets of axles. Now the horizontal axle w_1 is connected with the crank axle through the bevel wheels k_1 and k_2; it

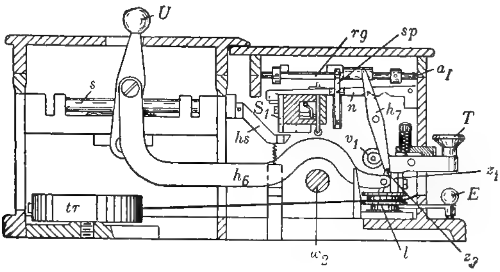


FIG. 43.—(⅓ actual size.) Side view (Section I I of fig. 42) to illustrate the reversing process.

carries two discs u, on which two rollers, the ends of a lever, move in such a manner that during a turn of the crank they execute an entirely constrained to-and-fro motion which is communicated through the lever connection h_1, h_2 (fig. 44) to the beam b. The action is such that during the first half of a crank turn the beam b is pressed upwards, the coupling established, and the forward motion of the wheels of the slot mechanism communicated to those of the carriage; but then, at the moment the former wheels cease to revolve before the next half of the turn, the beam is depressed and the coupling released during the return motion. On the beam being lowered a pin *st*

catches in a gap of the coupling wheels, so that they maintain their correct position until they are re-engaged. The to-and-fro movements communicated to the racks by the crank through the connecting rod are not uniform, but are quickened towards the middle of the crank turn, and fall off finally to zero. This circumstance is one of far-reaching importance in the whole construction of the machine. For the rotation of the axles in the slot and carriage mechanisms falls off simultaneously towards the end, so that the latter, on uncoupling, immediately stand still, and no kind of inertia effects can possibly appear. Therefore to secure the axles a_1 in their positions a catch d_1 is sufficient. This catch is pressed by a spring against a toothed wheel near the number cylinder and springs against it immediately a number appears in the viewhole P. The ends of the carriage axles project out of the machine: we can set up numbers in division, etc., by means of them. Special safeguards are provided here to prevent a rotation past 9, which would cause a carrying over of ten.

From what has been said, the number cylinders in P are rotated during the first half of the crank turn by the amount of the digits set up in the corresponding places on the slots (in subtractions it is their nines comple-

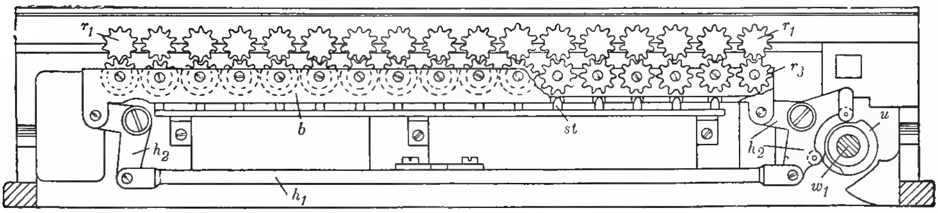


FIG. 44.—($\frac{1}{3}$ actual size.) Coupling as seen from above (Section III III of fig. 42).

ments); the second half of the crank turn is reserved for the completion of the process (which has been "prepared for" already) of carrying over the tens, and the raising of the next highest place in the passage from 9 to 0 in the carriage mechanism. This is carried out in the following manner. To the axis a_1 (figs. 45 and 46) there is freely attached a clutch m, with a disc p_2, from which projects a pin, passing through an opening in the disc p_1, this latter being rigidly attached to the axle. If the number cylinder in the viewhole P turns from 9 to 0, the pin thereby comes into contact with an attachment c on the machine frame, and is pushed along over its sloping surface so that the clutch is displaced along the axle. It is held firm in this new position by the spring catch i, lying behind the disc p_3. The completion of the process of carrying over the ten is effected from the axle w_2, which is coupled to the horizontal axle w_1 by the bevel wheels k_2. As the circumferences of these wheels are in the ratio 2 : 1, the axle w_2 makes two revolutions with one crank turn. On it are set spirally a number of eccentric pairs e_1, e_2, one pair under each carriage axle. Being linked to the lever h_1 (fig. 42), the axle, like the lever, is slightly displaced longitudinally at the beginning of the first revolution, but at its second revolution it is brought back to its old position, so that the eccentrics are now under the cams f_1, f_2 (figs. 45 and 46), and, instead of passing them as they did previously, they force them upwards by their further rotation. The cams f_1 now move over the

surfaces O of the fixed frame. If they experience no resistance, they rise perpendicularly and are then immediately drawn back to their initial position by the spring *fh*, after the eccentrics have passed by them. If, however, a process for carrying over a ten has been initiated, the corresponding cam f_1 strikes against the projecting flange *fl* of the clutch *m*, is tipped by it to the side, and with the tooth *y* advances by a unit the cog wheel on the neighbouring axle. This procedure is represented in fig. 46 by the highest cam. Meanwhile the eccentric e_2, which lags behind the previous one by a small amount, has elevated the cam f_2; this meets an arm of the catch *i*, releases it, and

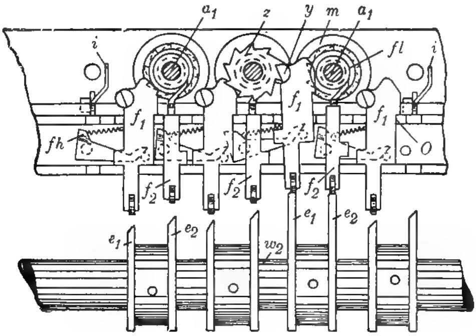


FIG. 45.—($\frac{2}{3}$ actual size.)

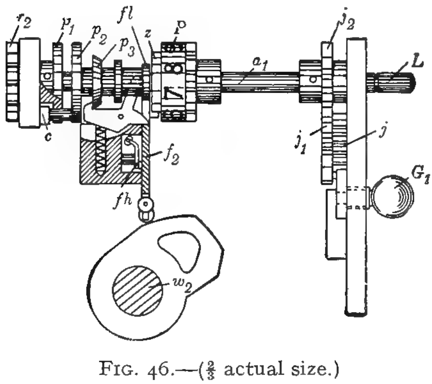


FIG. 46.—($\frac{2}{3}$ actual size.)

Mechanism for carriage of tens (front and side view).

pushes the clutch back by means of a lever into its initial position. The cam f_1 is thereby set free and falls according to the run of the eccentric. Since the eccentrics are arranged spirally, the carrying over of tens goes on continuously from the lowest place, and may proceed through the whole mechanism. The process of carrying over a ten can only take place during the second half of the calculation, when the coupling bar is off. The double rotation of the shaft w_2, however, makes it possible to spread the eccentrics over almost the whole periphery of the axle w_2, and to give them correspondingly smaller radii. After giving the preceding description it is unnecessary to emphasise the fact that all parts of the operation of carrying over tens are performed automatically, and therefore we get a safe guarantee that the action is free from error.

The number cylinders of the quotient, which indicates the number of crank revolutions in single positions of the carriage, and can be seen in the row of viewholes Q, are attached to cylindrical collars H_1 on the axles a_1, and in consequence of this arrangement (a very handy one for the calculator) appear in the same line with the markers and the carriage figures. This mechanism is driven from the axle w_3 (fig. 42), which is coupled by means of an intermediate wheel with the eccentric shaft w_2, and thereby also with the crank handle. This shaft can be displaced longitudinally and carries the two bevel wheels k_4 and k_5, which may in turn be engaged with k_3, and on its left end a gear wheel which drives the shaft w_4 higher up (fig. 47). With chosen adjustments of all these wheels, w_1 makes with one crank turn a revolution (direct or reverse, according as the wheel k_4 or k_5 is engaged). The reversing takes place by means of the reversing lever U_1 at one end of a lever; a rod *ss* (fig. 42) communicates the latter's motion to the lever h_3, which engages with a clutch on the shaft w_3, and displaces it to one side or the

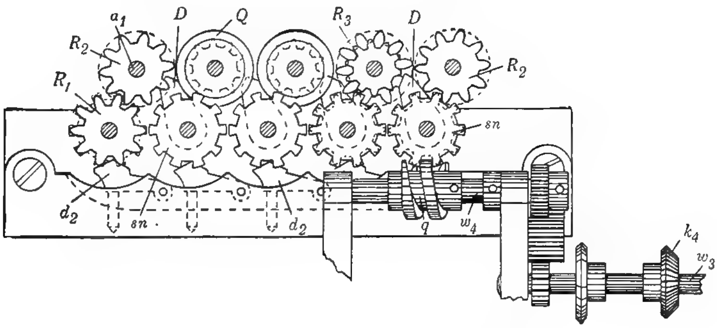


FIG. 47.—($\frac{2}{3}$ actual size.) Quotient mechanism.

other. A spring causes the reversing lever to spring easily into its end position, so that it is held firmly there. Similar precautions are taken as in the case of the lever U to prevent turning of the mechanism when the setting up is incorrect. Reversal during a calculation is likewise impossible.

The worm on the shaft w_4 drives the ten-toothed cog wheel *sn* above it a tooth further at every revolution. This, together with a cog wheel R_1, and a fixed projecting arm D, lies on a sleeve revolving on a fixed axle. Above this, finally, on the main carriage axles, are seated collars H_1, carrying the two cog wheels R_2 and R_3 near the number cylinders. These parts act in the following way:—The two toothed wheels R_1 and R_2 engage with each other (left of fig. 47). At every turn of the worm the number cylinder Q is advanced a unit. If thereby a passage from 9 to 0, or by reverse motion from 0 to 9, takes place, the arm D catches in the cog R_3 of the next highest place and advances or retracts it one digit. As in the product mechanism, springs d_2 press against the teeth of the wheels R_1, so that the correct position of the gear wheels and of the numbers in the viewholes is maintained. In order that a displacement of the carriage and its accompanying mechanism past the non-movable driving screw *q* may be possible, the latter is provided with a slot, which in normal positions of the crank lies in the plane of the cog wheels *sn*, and through which therefore they pass freely.

The "carrying over of tens" in the quotient is an outstanding feature of the new machine, and is of extreme importance in the process of "contracted multiplication." It is generally the custom with an addition machine to carry out the multiplication of a number of several digits (say 299, for example) so that it is multiplied by 300 and then one subtracted in the units place. The older machines, however, indicated as the multiplier a number 301 instead of 299, and the one was differently coloured to distinguish the subtraction part. It fell to the calculator, then, to carry this number in his head, to convince himself of the correctness of his operation. In the application of this method of calculating, it is only necessary with the "Euklid" to reverse both levers U and U_1 in subtraction, placing U on subtraction, U_1 on C, *i.e.* correction for the multiplier (fig. 40), and then to turn so many times, until the desired multiplier appears in Q.

The carrying over of tens in the quotient was absolutely necessary in automatic division (mentioned above), and the fundamental idea will be here briefly indicated, so that the mechanism required may be afterwards described in detail. Let the division of a number a by b give in the quotient the first two numbers c and d, and the corresponding remainders r_c and r_d; then we get the equation—

$$\frac{a}{b}=c\cdot 10^n+\frac{r_c}{b}=c\cdot 10^n+d\cdot 10^{n-1}+\frac{r_d}{b}, \quad . \quad . \quad (1)$$

or

$$\frac{a}{b}=(c+1)\cdot 10^n-(10-d)\cdot 10^{n-1}+\frac{r_d}{b}. \quad . \quad . \quad (2)$$

In equation (2) we are given the mathematical expression for the procedure in automatic division. Instead of subtracting the divisor at each place so many times from the dividend, till we get a positive remainder, which is smaller than the divisor—in the first place c times, in the second d—we carry out the subtraction $(c+1)$ times in the first place and get a negative remainder $\frac{r_c-b\cdot 10^n}{b}$, to which we add in the next place so many times, until the remainder is again positive, that is, according to equation (2), $(10-d)$ times. The same process is repeated in the third and fourth places, and so on. In the carrying out of such divisions with our calculating machine, after setting up the dividend and divisor, we displace the carriage until we bring their highest places opposite each other, place the lever U on subtraction, U_1 on N (*i.e.* normal position or addition of the crank turns), and then turn the crank so many times—$(c+1)$—until the dividend is negative, which is indicated by a number of nines to the left of the carriage axles. In the mechanism we now get a self-acting check, which is only removed when both levers are reversed and U placed on addition, U_1 on C (correction for quotient), whereupon the carriage moves one place to the left. We now turn $(10-d)$ times, and get, on account of the carrying over of tens on, the quotient, its correct value cd in Q; during the last turn the dividend again becomes positive, and we get a check. Only on reversing again can we proceed, when the process just described is repeated. We see from this that the machine must be provided with a contrivance for advancing the carriage one place

automatically on reversal; further, we must get a check on the crank if either nines appear on the left of the carriage in subtraction, or the nines change to nothings in addition.

The arrangements for automatic displacement of the carriage are represented in figs. 42, 43, 48, and 49. The carriage runs on rollers supported by guides in the frame. To it is attached a linked chain passing round a pulley l, and pulled by a strong spiral spring lying in the drum tr, so that the carriage is constantly drawn towards the left. Fixed to the base of the machine is a rack z_1, into which engages a projection V on the key T of the carriage. The teeth of the rack z_1 are sloped (fig. 48) on one side, so that the projection can move over them without resistance or displacement of the carriage to the right, while on the return motion it is pressed against their perpendicular side by a spring. A pressure on the key T removes the check, and the carriage can be pushed into any other required position, remaining there when the key is released. The distances between the teeth are equal to the distances between the axles of the carriage and slot mechanisms, and the key is so constructed that in every position of the carriage the cogs r_1

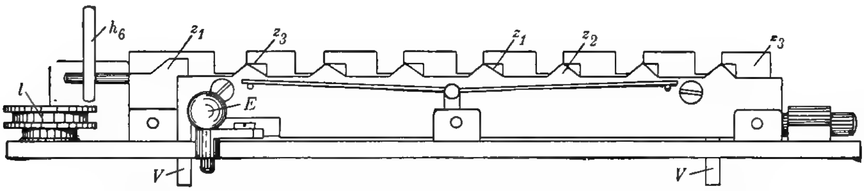


FIG 48.—($\frac{2}{3}$ actual size.) Displacement of the carriage (front view).

and r_2 on these axles are opposite each other. In multiplication we require an automatic displacement of the carriage from place to place; we use for this purpose a knob K_n on the cover of the slot mechanism, which can be placed, if required, to the left of the crank K, so that the displacement of the carriage during a calculation can be made easily with the thumb of the right hand, without letting go the crank. A quick pressure on this knob is transferred by the lever h_4 to the arms h_5 (fig. 42), the sloped teeth of which press against the projections V of a second rack z_2, placed in front of z_1, and displaceable vertically. These are raised up, and the projecting piece V is thereby disengaged from the rack z_1. The carriage is then displaced so far until V meets the vertical side of a tooth of the rack z_1, when it stops at the next place. z_2 meanwhile has returned to its former position under the action of two springs. In automatic division the displacement of the carriage must follow automatically on reversal of the lever U. This is effected by a swinging rack z_3, worked by the lever h_6 from the reversing lever, which acts on the roller U_1, and releases T. This rack has openings corresponding to those in the rack z_1. If, on reversing, the key T is released, the carriage moves to the left until the roller springs into one of these openings and prevents further motion. After the rack has swung out to its fullest extent, the projection V can engage in the next hole, and complete displacement is got. As it is not desirable in every kind of calculation to have the carriage automatically displaced, a contrivance for longitudinal displacement of z_3 is provided, which causes the

roller v_1 to face the openings, thereby preventing the lateral motion of the rack having any action on the roller. This longitudinal displacement is effected by a lever E, which can be put in either of the two previously described positions (figs. 48 and 49). To guard against displacement of the carriage

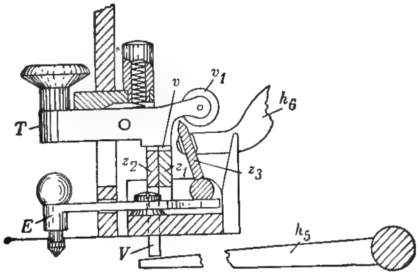


FIG. 49.—($\frac{2}{3}$ actual size.) Displacement of carriage (side view).

during a calculation, and also to prevent turning the crank in an incorrect position of the carriage, there is attached to the frame of the slot mechanism, underneath the crank axle, a lever *hs* (fig. 43). A roller at one end of it is pressed by a spring against a disc p_4 on the crank axle, and springs into a notch of p_4 in the normal position of the crank. The other end of this lever is fitted with a projecting piece, which faces a rail S fixed to the frame of the carriage. This rail is fitted with notches, at distances from each other equal to those of the carriage axles, into which the projection engages in the correct

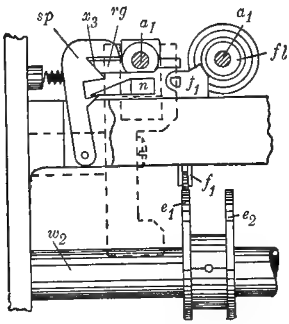


FIG. 50.—($\frac{2}{3}$ actual size.) Last carriage axle with fittings for automatic check (front view).

position of the carriage, if a turn of the crank presses the lever *hs* downwards from the disc p_4. A displacement of the carriage during a crank turn is thus made impossible. If the carriage is incorrectly displaced, the crank is prevented from turning, since the lever strikes against the rail S.

At the same time the lever *hs* serves as a brake on the crank in automatic division. For this purpose there lies alongside S a second rail S_1, fitted with sloped teeth, which is displaced slightly in its longitudinal direction at each crank turn by the projection, which is likewise fitted with a sloping surface. The check now takes place in the following way :—The carriage axle, lying to the extreme left, is provided, like the others, with all the arrangements for carrying over tens. To the left of it is an auxiliary axle fitted with a bolt

rg (figs. 50 and 51), which can turn round the axis or be displaced along it; it is displaced on reversal from U by a lever h_7 (fig. 43) attached to the swinging rack z_3; in subtraction taking up the position of figs. 43 and 51; in addition, on the other hand, coming nearer the forepart of the carriage. In carrying out a division, the divisor is subtracted as many times as it is contained in the corresponding place in the dividend. As the machine does this by adding the tens complements (compare the example on p. 107), there appear first in the higher places of the carriage a number of nines, which become nothings on carrying over ten. If this continues up to the highest place, a process for carrying over ten will also be initiated here, and the flange *fl* will strike against the cam f_1, which is here fitted with two small projections. On being elevated by the excentric e_1, this is tipped slightly to the left, and passes

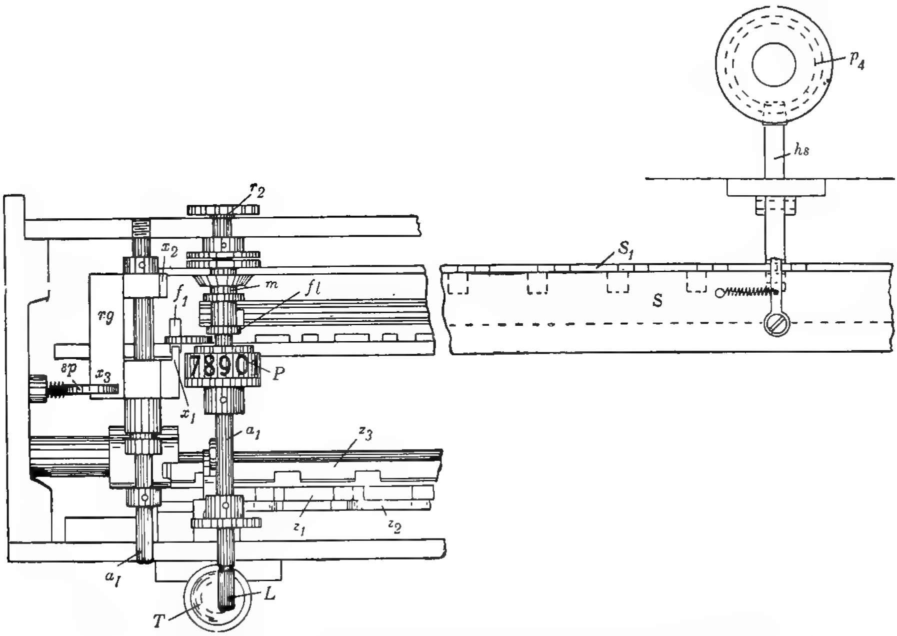


FIG. 51.—($\frac{2}{3}$ actual size.) Last carriage axle with the fittings for automatic check (appearance from above).

without touching the projection x_1 of the bolt *rg*. This takes place at every turn, as long as the dividend is still positive; but if a still further subtraction of the divisor is carried out, then the nines remain in the carriage mechanism, no ten is carried, and the cam rises vertically, meets the bolt at x_2, and tips it round, as shown in fig. 51. This procedure is reversed in the second part of automatic division, the addition of the divisor to the next lowest place in order to correct the quotient. The surface x_2 of the bolt then faces the cam and is not touched by it, as long as there is no ten carried over. As soon, however, as the negative dividend again becomes positive by adding the divisor to it sufficiently often, in place of nines, nothings appear again with the progressive carrying over of ten; the projecting flange *fl* now thrusts the cam f_1 aside, and this latter tips the bolt round at x_2. A check to the crank is thereby got in both cases. For the bolt *rg*, on being tipped round,

presses with its sloping surface x_3 the hook *sp* against a spring. This releases a swing lever *n*, which is pivoted to the forepart of the machine, from a small projection of the hook, and inserts it by means of a spring in an opening of the movable rail S_1 (fig. 43). This is thereby secured against longitudinal displacement; the lever *hs* in consequence remains immovable, and so checks the turning of the crank. The removal of the check takes place during reversal; the lever h_7 through its motion raises the bar *n* and replaces it in its initial position, in which it is held fast by the hook *sp*. If a pull on the crank were to be transferred to the mechanism after a check had been imposed, then injurious results would easily follow improper usage. To prevent this, the crank is constructed in a special way. To the crank shaft is fixed a disc t_1, and above it a rotary disc t_2, to which is attached the crank K (fig. 52); between them is placed a spiral spring which takes up the strain on the crank and carries it over to the axle. With a greater resistance in the mechanism it is contracted, and a pin is pressed by a sloping surface inside t_2 into a depres-

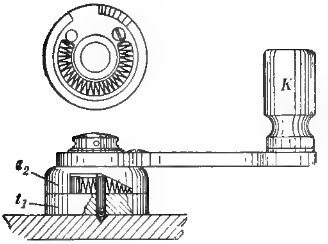


FIG. 52.—($\frac{1}{2}$ actual size.) Construction of the crank.

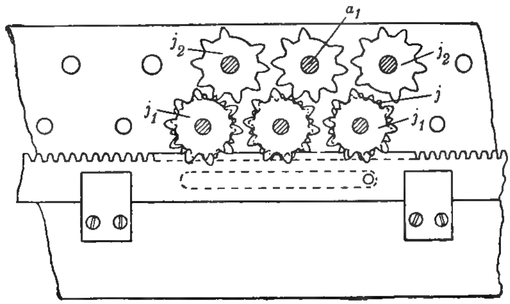


FIG. 53.—($\frac{2}{3}$ actual size.) Effacer.

sion in the top of the machine. This then takes up any further strain on the crank, and possible injurious effects are avoided. A reverse turn of the crank, which the internal construction of the machine will not allow, is prevented by a pin which is pressed by a spring against the discs *p* (fig. 42). On turning the crank in the right direction, it is pushed back; on reversing the crank, it falls between these discs and keeps them immovable. To keep the crank in its normal position there is also provided a spring lever *h* (fig. 42). whose rotating end carries a roller, which fits into a depression in the axle W_1, when the crank takes up its initial position.

The last essential part of the machine which requires mention is the "effacer." This is put in action for each of the product and quotient mechanisms by pulling aside the knobs G and G_1. A rack and pinions *j* (fig. 53) engaging with it are thereby set in motion. The axles of these pinions carry in addition a ten-toothed wheel j_1, which engages with the wheel j_2 on the axle a_1 of the product mechanism. A tooth is absent in both, so that in certain positions we have a gap between them. When the rack is not in motion, the hole in j_1 is opposite the toothed wheel j_2, which can then move freely. On being displaced, however, j_1 engages with the toothed wheel j_2 and rotates it until its hole comes underneath, when contact with j_1 ceases. All the number cylinders are simultaneously put back to zero. Pulling on the knob G similarly effaces the quotient or multiplier Q. Both knobs are then

brought back to their initial position by means of springs; G at the same time can be used as a handle to pull back the carriage to its normal position. In conclusion, it may also be mentioned that all parts of the machine which have stronger demands made on them, such as the main axles, the racks for displacement of the carriage, the eccentrics, etc., are made of hardened steel, so as to ensure durability. Further, we must refer to the fact that the machine permits of an extended use by the provision of a second slot mechanism in front of the carriage mechanism. Thus products of the form $a \times b \times c$ can be formed, without necessitating a new setting up of the product $a \times b$, and in the adding of simple products not only the sum but the simple products can be read off. In general, the most involved calculations can be easily and quickly carried out. Improved machines are also in process of construction, and will shortly be put on the market.

It is astonishing with what speed and accuracy the machine completes all kinds of calculation, and especially automatic division. The striking innovations introduced into the Euklid, opening up entirely new fields to machine calculation, will assure it a prominent place among mechanical aids to calculation.

(9) **The "Millionaire" Calculating Machine.** O. Steiger, Patentee.

This machine is used for working out all calculations which can be made by the four rules of arithmetic. Its principal advantage consists in the simplicity and rapidity with which multiplications, divisions, square roots, and compound rules may be treated.

For each figure of the multiplier or quotient only *one* rotation of the crank is necessary, while the displacement of the product takes place simultaneously and automatically.

In the representation of the machine in fig. 1 there may be distinguished:—

The *regulator* U, by means of which the machine is adjusted for the different kinds of calculations. It is placed in the position marked A, M, D, S (Addition, Multiplication, Division, Subtraction), according to the calculation required.

The *crank* K, which is turned once in the direction of the arrow for each figure in the multiplier or quotient, or for every addition or subtraction.

The *multiplication lever* H, which is in one of the positions 0 to 9 according to the multiplier or quotient. (For additions or subtractions it is placed on "1.")

The *markers* "*e—e.*"—The amount to be added or subtracted, the multiplier or divisor are placed in position by sliding the knobs down the vertical rows of figures until the points are opposite the figures required; the *control dials* "e^1—e^1" form a valuable check, since they repeat in a straight line the numbers recorded by the markers "*e—e.*"

Row of control dials "*f—f,*" which show automatically the multiplier or the quotient while the crank is being turned.

Row of result dials "*g—g,*" which register the amount, remainder, product, or dividend. The numbers may also be placed by hand by turning the knobs of the dials.

Effacer of result numbers R
Effacer of control numbers C } These knobs are drawn to the ends of their slots and then brought back *gently* to their former positions.

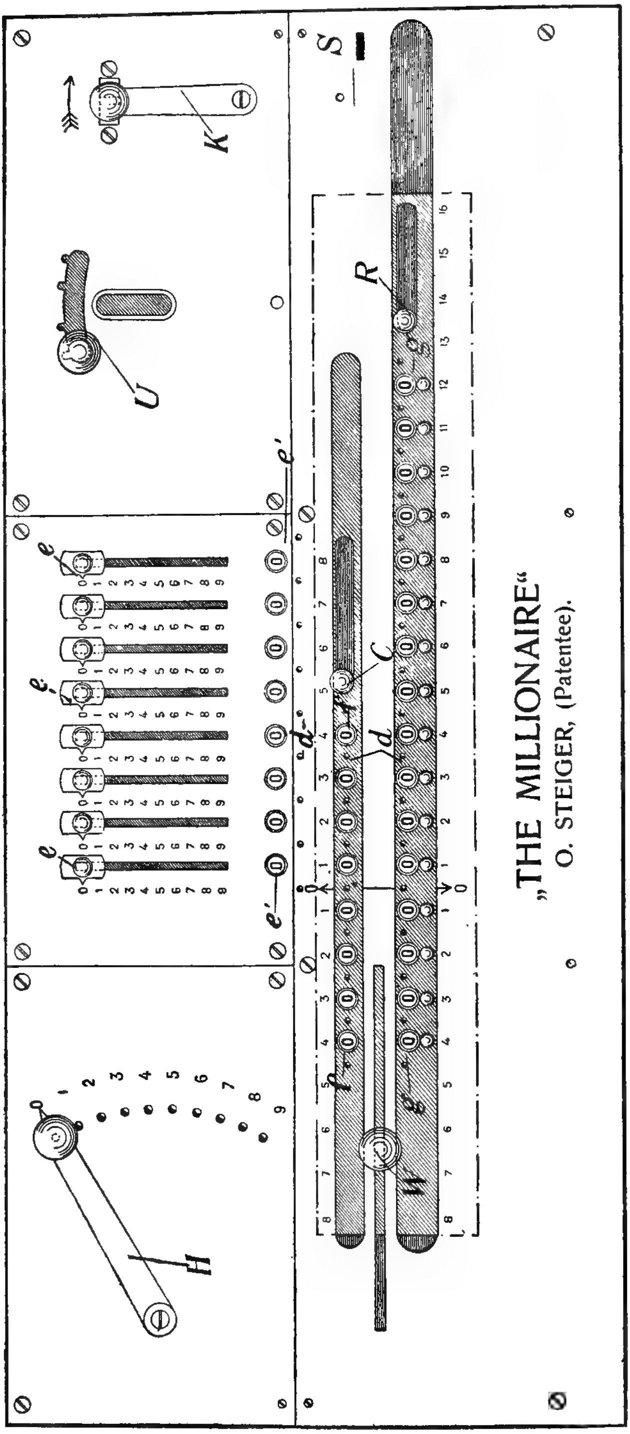


FIG. 54.—"Millionaire" Calculating Machine.

Carriage shifter W, which serves to place the registering part of the apparatus (hereafter called the "recorder"), comprising the result and control dials, in one of the eight possible positions.

The "Millionaire" calculating machine is a true multiplying machine, while the other systems of calculating machines in use are only addition machines, and as such carry out multiplication by a series of additions. (Subtractions and divisions may be regarded as additions and multiplications in the negative sense, and are therefore not further considered.) Clearly a multiplying machine which can only be used for the multiplying digit "1" is merely an addition machine.

In the "Millionaire" calculating machine are comprised three principal pieces of mechanism (see figs. 55, 56, and 57):—

(1) The multiplying mechanism.
(2) The carrying mechanism.
(3) The recorder, which is itself divisible into two parts, whereof one (viz., *g—g*) registers the product, while the second (*f—f*) is only for convenience, since it indicates the multiplier, but as such is not absolutely essential to the multiplying machine.

The *multiplying mechanism* consists of the so-called multiplying pieces and their supporting mechanism, which permits of motion:

(1) in the vertical direction;
(2) in the horizontal direction lengthwise;
(3) in the horizontal direction diagonally.

The multiplying pieces, which form the most essential part of the machine, consist of (fig. 55) nine tongue-plates, of which

the first gives the products of 1 to 9 times the number 1,
the second gives the products of 1 to 9 times the number 2,

and so on, the ninth the product of 1 to 9 times the number 9, so that the whole multiplication table is represented. Each of these products is expressed by two elements (tongues), of which one gives rise to the tens and the other to the units.

All the tens of a tongue-plate form a group by themselves, as also the whole of the units, and these groups act one after another, with the carrying mechanism and the recorder.

An inspection of fig. 55 shows each individual product; thus on plate 7 for the factor 6 we have 4 tens and 2 units, the product $7 \times 6 = 42$.

The *carrying mechanism* consists of:—

(*a*) Nine parallel toothed racks Z.
(*b*) The transverse axes, along which the pinions T are displaced by the knobs *e* on the indicating plate of the machine, and are thereby caused to engage with any one of the nine toothed racks, corresponding with a given position of the multiplicand.

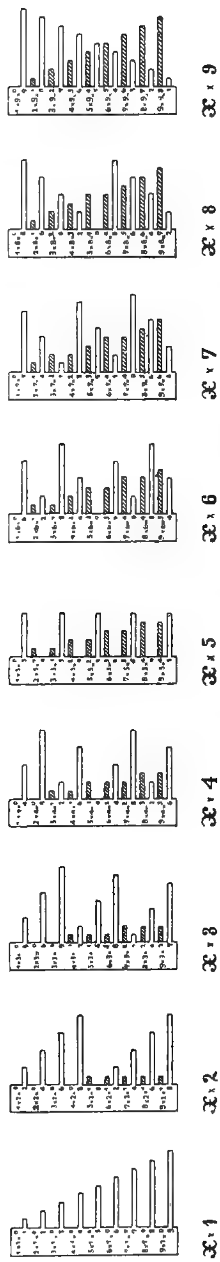
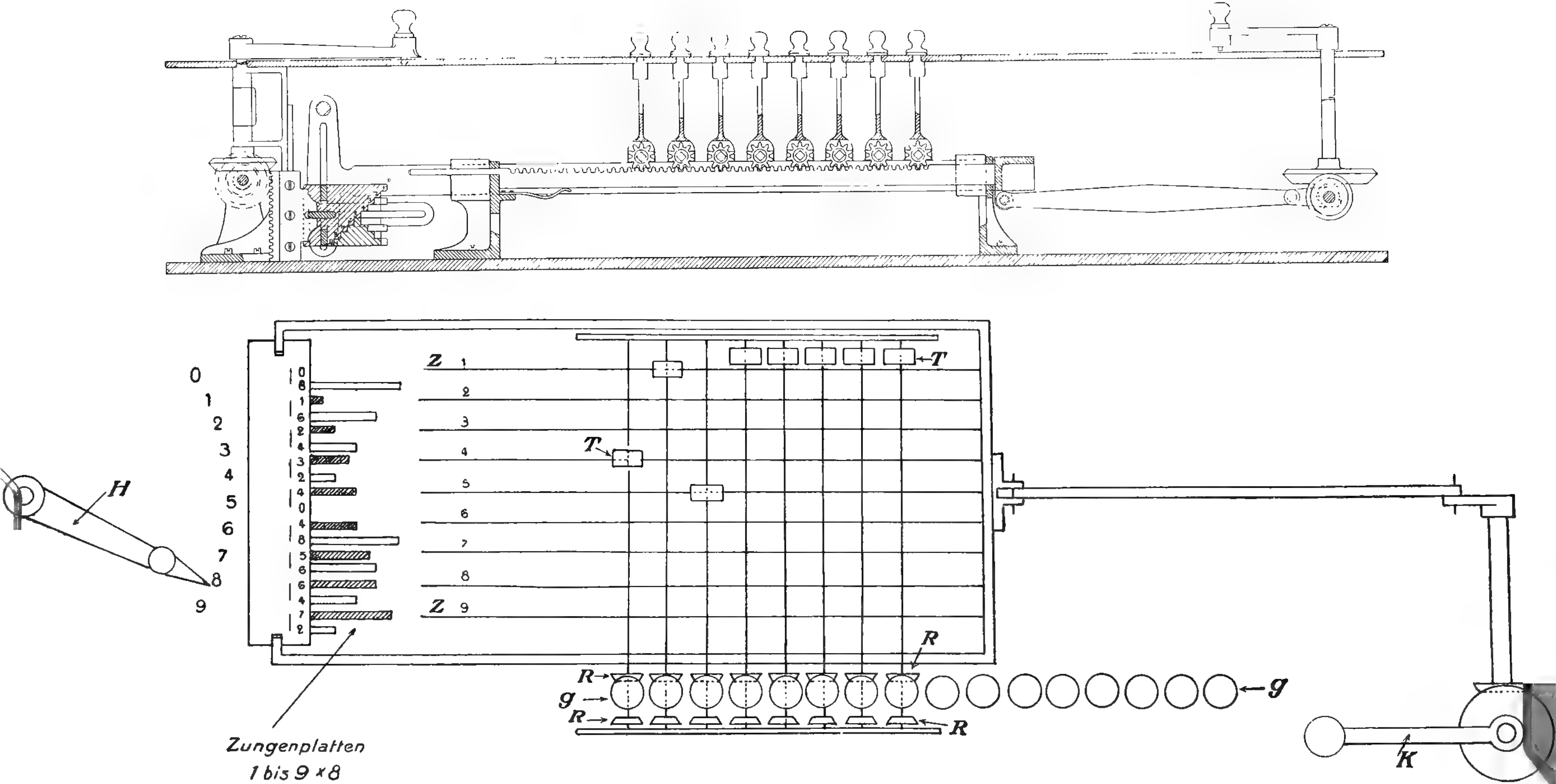


FIG. 55.—The Tongue-plates of the Multiplying Pieces for the Factors 1 to 9.

On each of these axes is a pair of bevel wheels R, which can be moved along the axis. They transfer to the recorder the rotations of the pinions T, which correspond to the longitudinal motion of the racks.



FIGS. 56 and 57.—Mechanism of the "Millionaire" Calculating Machine.

By means of corresponding mechanisms for inward and outward movements these bevel wheels are periodically engaged and disengaged with the recorder, so that the latter is influenced only during the forward displacement of the toothed racks.

The ends of the racks rest against either the tens or the units group of the tongues of a tongue-plate. The change of the groups is accomplished through the small horizontal diagonal displacement of the multiplying pieces, while the adjustment of the various tongue-plates is secured by the movement of the lever H over a scale. By each turn of the crank K, *i.e.* by multiplication by a given factor, the racks are displaced first to the tens and then to the units.

Since the tens and units of the multiplying pieces are represented by equal length-units, it is necessary, after carrying over the tens-value, to displace the recorder one place to the left, so that the units-value is registered one place to the right of its ten-value.

The action of the calculating machine is thus explained. To make it clearer, an actual example will be taken.

Let it be desired, for instance, to multiply 516 by 8. Then by displacement of

the 1st	knob *e* (from the left)	the pinion T	is moved	to	the	rack	5
,, 2nd	,,	,,	,,	,,	,,	,,	1
,, 3rd	,,	,,	,,	,,	,,	,,	6

The multiplier is then set on the number 8 of the scale by the lever H, whereby the tongue-plate *x* by 8 is placed against the racks. During one rotation of the crank K the multiplying-piece is twice thrust against the racks Z, and these are displaced corresponding to the tens and units of the product of 1 to 9 times 8.

In our case, by means of the racks	5	1	6	
the products	$5\times8=40$	$1\times8=08$	$6\times8=48$	
are carried over, so that the apparatus first registers the tens . .	4	0	4	
to which, after these have been moved one place to the left, units are added		0	8	8
to obtain			$\leftarrow$ 1	
the product	4	1	2	8

For every rotation of one of the figure-dials of the recorder in the positive or negative sense above 0 (or 10) ±1 is added to the next left-hand dial.

The following summary shows the sequence of the various operations in the calculating machine during one rotation of the crank :—

Rotation of the crank K from 0°–360°.	
0°	Coupling of the bevel wheels of the carrying mechanism with the recorder.
0°–90°	Carrying over of the tens and addition to the amount already recorded, giving the tens.
⟶	Uncoupling of the bevel wheels from the recorder.
90°–180°	Idle return-stroke of the racks. Displacement of the recorder to the left. Carrying over of the tens resulting from the addition.
⟶	Coupling of the bevel wheels with the recorder. Diagonal displacement of the multiplying pieces.
180°–270°	Carrying of the units and addition to the tens already obtained.
⟶	Uncoupling of the bevel wheels from the recorder.
270°–360°	Idle return-stroke of the racks and carrying of the tens obtained by addition.
⟶	Diagonal displacement of the multiplying pieces to their original position.

The construction of the "Millionaire" calculating machine is strong and reliable. The machine has been on the market for fifteen years, and as early as 1912 there were over two thousand in use.

Examples to illustrate Speed

(*a*) Multiplications:

$350{\cdot}729 \times 357$	$=125210{\cdot}253$	in 2 or 3 seconds.
18769423×23769814	$=446145693597322$	,, 6 ,, 7 ,,
$716^2 \times 535^2$	$=798881$	,, 8 ,, 9 ,,

(*b*) Eight factors; leading digits:

$$\left.\begin{array}{r} 125 \times 37572 \\ 4212 \times 8014 \\ 9 \times 277 \\ 50803 \times 7899 \end{array}\right\} = 439746858 \text{ in 30 or 35 seconds.}$$

(10) **The Thomas de Colmar Arithmometer.**

The first machine to perform multiplication by means of successive additions was that of Leibnitz, which was designed in 1671 and completed in 1694. It employed the principle of the "stepped reckoner." This model was kept first at Göttingen and afterwards at Hanover, but it did not act efficiently, as the gear was not cut with sufficient accuracy. This was long before the days of accurate machine tools.

The first satisfactory arithmometer of this nature was that of C. X. Thomas, which was brought out about 1820. It is usually called the Thomas de Colmar Arithmometer. It is still a useful machine, but its place is now being taken by lighter and better types.

The fundamental principle of the mechanism is illustrated in the diagram. C is the carriage, which, when raised, may slide and turn about a horizontal axis. It carries on its face the product holes, and the multiplier holes, with their indicators, and also two milled heads M, which engage with racks and springs for clearing the digits.

On the body of the machine there are from six to ten slots bearing on their edges the multiplicand digits, with studs S, which are set to the required values.

Any stud S shifts, by sliding, the pinion B′ along its axis *b*, so as to engage with the requisite number of the unequal teeth on the barrel of the stepped reckoner A. The cross-section of the axis *b* is square. H is the handle by which the machine is worked. It rotates the vertical spindle shown, and the pair of bevel wheels at its base drive the stepped reckoner A. Thus B′ for

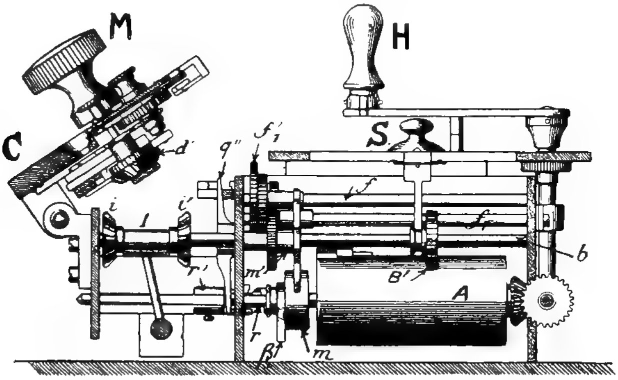


FIG. 58.—L. Jacob, *Le Calcul Mécanique*. (Doin, Paris).

one revolution of H gives a rotation to *b* corresponding to the digit at which S is set.

If the carriage C is lowered so that the bevel wheels *d′* and *i′* engage, this rotation is conveyed through *d′* to the indicators of the product holes, where the result appears. Multiplication is thus performed by successive additions.

For subtraction the sleeve I is pulled by a small lever along the axis of the shaft *b*, so that the other edge of *d′* engages with *i*, and thus a negative rotation is communicated to the indicators of the corresponding product holes. Division is effected by successive subtractions.

For the carrying device there is a cam on the spindle of the number wheel of the product indicator in the sliding carriage. As the indicator number changes from 9 to (1) 0, a pin on this cam shifts a lever in the body of the machine. This moves a sliding piece which, by a suitable arrangement, rotates the next indicator axle by one tooth and so produces the required result.

In some of the recent forms of Thomas Arithmometer there are twenty product holes.

The Tate Arithmometer is similar in construction to the Thomas. See *Die Thomas'sche Rechenmaschine*, by F. Reuleaux, 2nd ed., Leipzig, 1892.

II. Automatic Calculating Machines. By P. E. Ludgate.

Automatic calculating machines on being actuated, if necessary, by uniform motive power, and supplied with numbers on which to operate, will compute correct results without requiring any further attention. Of course many adding machines, and possibly a few multiplying machines, belong to this category; but it is not to them, but to machines of far greater power, that this article refers. On the other hand, tide-predicting machines and other instruments that work on geometrical principles will not be considered here, because they do not operate arithmetically. It must be admitted, however, that the true automatic calculating machine belongs to a possible rather than an actual class; for, though several were designed and a few constructed, the writer is not aware of any machine in use at the present time that can determine numerical values of complicated formulæ without the assistance of an operator.

The first great automatic calculating machine was invented by Charles Babbage. He called it a "difference-engine," and commenced to construct it about the year 1822. The work was continued during the following twenty years, the Government contributing about £17,000 to defray its cost, and Babbage himself a further sum of about £6000. At the end of that time the construction of the engine, though nearly finished, was unfortunately abandoned owing to some misunderstanding with the Government. A portion of this engine is exhibited in South Kensington Museum, along with other examples of Babbage's work. If the engine had been finished, it would have contained seven columns of wheels, with twenty wheels in each column (for computing with six orders of differences), and also a contrivance for stereotyping the tables calculated by it. A machine of this kind will calculate a sequence of tabular numbers automatically when its figure-wheels are first set to correct initial values.

Inspired by Babbage's work, Scheutz of Stockholm made a difference-engine, which was exhibited in England in 1854, and subsequently acquired for Dudley Observatory, Albany, U.S.A. Scheutz's engine had mechanism for calculating with four orders of differences of sixteen figures each, and for stereotyping its results; but as it was only suitable for calculating tables having small tabular intervals, its utility was limited. A duplicate of this engine was constructed for the Registrar General's Office, London.

In 1848 Babbage commenced the drawings of an improved difference-engine, and though he subsequently completed the drawings, the improved engine was not made.

Babbage began to design his "analytical engine" in 1833, and he put together a small portion of it shortly before his death in 1871. This engine was to be capable of evaluating any algebraic formula, of which a numerical solution is possible, for any given values of the variables. The formula it is desired to evaluate would be communicated to the engine by two sets of perforated cards similar to those used in the Jacquard loom. These cards would cause the engine automatically to operate on the numerical data placed in it, in such a way as to produce the correct result. The mechanism of this

engine may be divided into three main sections, designated the "Jacquard apparatus," the "mill," and the "store." Of these the Jacquard apparatus would control the action of both mill and store, and indeed of the whole engine.

The store was to consist of a large number of vertical columns of wheels, every wheel having the nine digits and zero marked on its periphery. These columns of wheels Babbage termed "variables," because the number registered on any column could be varied by rotating the wheels on that column. It is important to notice that the variables could not perform any arithmetical operation, but were merely passive registering contrivances, corresponding to the pen and paper of the human computer. Babbage originally intended the store to have a thousand variables, each consisting of fifty wheels, which would give it capacity for a thousand fifty-figure numbers. He numbered the variables consecutively, and represented them by the symbols V_1, V_2, V_3, V_4 . . . . . V_{1000}. Now, if a number, say 3·14159, were placed on the 10th variable, by turning the wheels until the number appeared in front, reading from top to bottom, we may express the fact by the equation $V_{10}=3{\cdot}14159$ or $V_{10}=\pi$. We may equate the symbol of the variable either to the actual number the variable contains, or to the algebraic equivalent of that number. Moreover, in theoretical work it is often convenient to use literal instead of numerical indices for the letters V, and therefore $V_n=ab$ means that the nth variable registers the numerical value of the product of a and b.

The mill was designed for the purpose of executing all four arithmetical operations. If V_n and V_m were any two variables, whose sum, difference, product, or quotient was required, the numbers they represent would first be automatically transferred to the mill, and then submitted to the requisite operation. Finally, the result of the operation would be transferred from mill to store, being there placed on the variable (which we will represent by V_z) destined to receive it. Consequently the four fundamental operations of the machine may be written as follows :—

$$
\begin{aligned}
&(1)\ V_n+V_m=V_z.\\
&(2)\ V_n-V_m=V_z.\\
&(3)\ V_n\times V_m=V_z.\\
&(4)\ V_n\div V_m=V_z.
\end{aligned}
$$

Where n, m, and z may be any positive integers, not exceeding the total number of variables, n and m being unequal.

One set of Jacquard cards, called "directive cards," (also called "variable cards") would control the store, and the other set, called "operation cards," would control the mill. The directive cards were to be numbered like the variables, and every variable was to have a supply of cards corresponding to it. These cards were so designed that when one of them entered the engine it would cause the Jacquard apparatus to put the corresponding variable into gear. In like manner every operation card (of which only four kinds were required) would be marked with the sign of the particular operation it could cause the mill to perform. Therefore, if a directive card bearing the number 16 (say) were to enter the engine, it would cause the

number on V_{16} to be transferred to the mill or *vice versa*; and an operation card marked with the sign $\div$ would, on entering the engine, cause the mill to divide one of the numbers transferred to it by the other. It will be observed that the choice of a directive card would be represented in the notation by the substitution of a numerical for a literal index of a V; or, in other words, the substitution of an integer for one of the indices n, m, and z in the foregoing four examples. Therefore three directive cards strung together would give definite values to n, m, and z, and one operation card would determine the nature of the arithmetical operation, so that four cards in all would suffice to guide the machine to select the two proper variables to be operated on, to subject the numbers they register to the desired operation, and to place the result on a third variable. If the directive cards were numbered 5, 7, and 3, and the operation card marked $+$, the result would be $V_5+V_7=V_3$.

As a further illustration, suppose the directive cards are strung together so as to give the following successive values to n, m, and z:—

Sequence of values for	n	. . 2, 6, 4, 7.
,, ,,	m	. . 3, 1, 5, 8.
,, ,,	z	. 6, 7, 8, 9.

Let the sequence of operation cards be

$$+ \quad \times \quad - \quad \div$$

When the cards are placed in the engine, the following results are obtained in succession:—

1st	operation,	$V_2+V_3=V_6$.
2nd	,,	$V_6\times V_1=V_7$.
3rd	,,	$V_4-V_5=V_8$.
4th	,,	$V_7\div V_8=V_9$.

From an inspection of the foregoing it appears that V_1, V_2, V_3, V_4, and V_5 are independent variables, while V_6, V_7, V_8, and V_9 have their values calculated by the engine, and therefore the former set must contain the data of the calculation.

Let $V_1=a$, $V_2=b$, $V_3=c$, $V_4=d$, and $V_5=e$, then we have

1st	operation,	$V_2+V_3=b+c=V_6$.
2nd	,,	$V_6\times V_1=(b+c)a=V_7$.
3rd	,,	$V_4-V_5=d-e=V_8$.
4th	,,	$V_7\div V_8=\dfrac{(b+c)a}{d-e}=V_9$.

Consequently, whatever numerical values of a, b, c, d, and e are placed on variables V_1 to V_5 respectively, the corresponding value of $\dfrac{a(b+c)}{d-e}$ will be found on V_9, when all the cards have passed through the machine. Moreover, the same set of cards may be used any number of times for different calculations by the same formula.

In the foregoing very simple example the algebraic formula is deduced from a given sequence of cards. It illustrates the converse of the practical procedure, which is to arrange the cards to interpret a given formula, and it also shows that the cards constitute a mathematical notation in themselves.

Seven years after Babbage died a Committee of the British Association appointed to consider the advisability and to estimate the expense of constructing the analytical engine reported that: "We have come to the conclusion that in the present state of the design it is not possible for us to form any reasonable estimate of its cost or its strength and durability." In 1906 Charles Babbage's son, Major-General H. P. Babbage, completed the part of the engine known as the "mill," and a table of twenty-five multiples of π, to twenty-nine figures, was published as a specimen of its work, in the *Monthly Notices of the Royal Astronomical Society*, April 1910.

I have myself designed an analytical machine, on different lines from Babbage's, to work with 192 variables of 20 figures each. A short account of it appeared in the *Scientific Proceedings, Royal Dublin Society*, April 1909. Complete descriptive drawings of the machine exist, as well as a description in manuscript, but I have not been able to take any steps to have the machine constructed.

The most pleasing characteristic of a difference-engine made on Babbage's principle is the simplicity of its action, the differences being added together in unvarying sequence; but notwithstanding its simple action, its structure is complicated by a large amount of adding mechanism—a complete set of adding wheels with carrying gear being required for the tabular number, and every order of difference except the highest order. On the other hand, while the best feature of the analytical engine or machine is the Jacquard apparatus (which, without being itself complicated, may be made a powerful instrument for interpreting mathematical formulæ), its weakness lies in the diversity of movements the Jacquard apparatus must control. Impressed by these facts, and with the desirability of reducing the expense of construction, I designed a second machine in which are combined the best principles of both the analytical and difference types, and from which are excluded their more expensive characteristics. By using a Jacquard I found it possible to eliminate the redundancy of parts hitherto found in difference-engines, while retaining the native symmetry of structure and harmony of action of machines of that class. My second machine, of which the design is on the point of completion, will contain but *one* set of adding wheels, and its movements will have a rhythm resembling that of the Jacquard loom itself. It is primarily intended to be used as a difference-machine, the number of orders of differences being sixteen. Moreover, the machine will also have the power of automatically evaluating a wide range of miscellaneous formulæ.

(1) H.M. Nautical Almanac Office Anti-Differencing Machine.

By T. C. Hudson.

This machine embodies successive developments (suitable for mathematical purposes) from the original Burroughs Adding-Machine of the years 1882–1891. It will work either in decimals, or in hours (or in degrees), minutes,

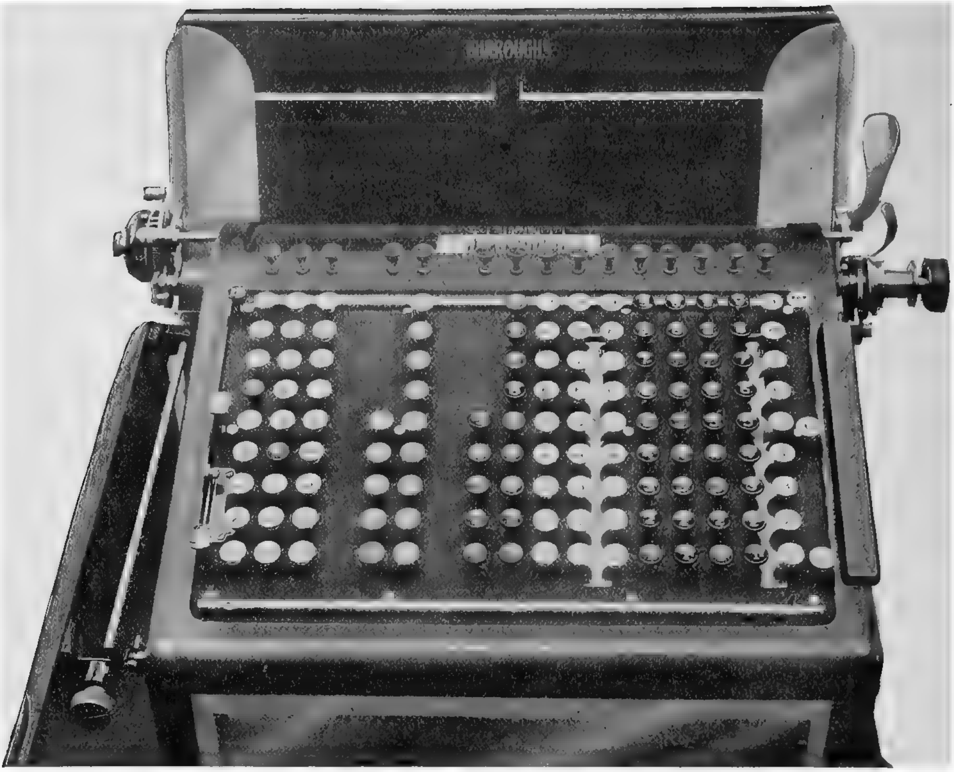


FIG. 1.—The Keyboard.

FIG. 2.—The Keyboard, showing the Multiplying Device.

seconds, and fractions. Its full capacity is shown by the figures 999^h 59^m $59^s{\cdot}999\ 9999\ 9$. Within these limits it will work to any degree of accuracy required, great or small. It will also record the result either to that same degree of accuracy (number of figures) or to any lesser degree. Thus, the machine may allow for a greater number of digits than it is required to record

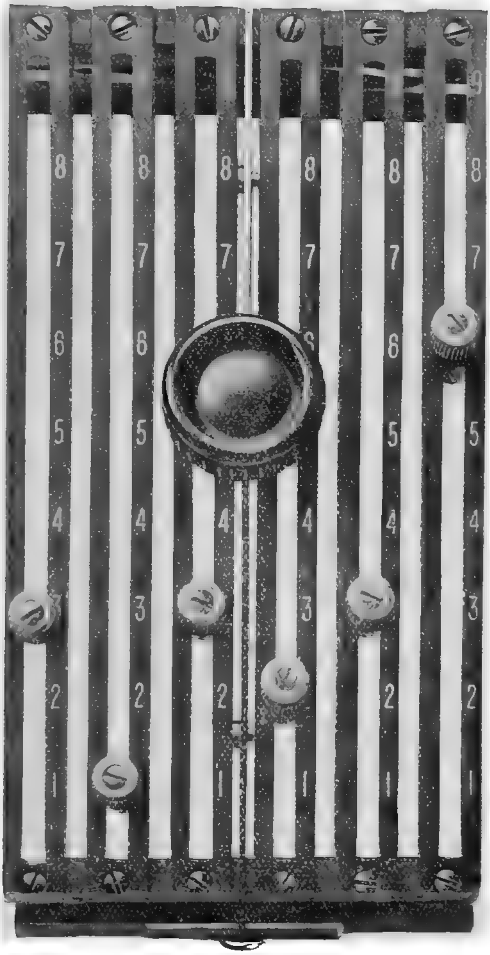


FIG. 3.—The Multiplying Device.

in the result. This feature is of obvious utility in table-making. The machine will *subtract* as well as add.

In particular, the machine fulfils the special purpose for which it was designed, namely, the production of serial quantities (for example, ephemeris quantities), of which every eighth, tenth, or twelfth (as the case may be) has been previously computed in full, but the last digit, *only*, of the seven, nine, or eleven intermediate quantities found accurately in groups by a pair of "graphs." Examples occurring in practice are:

The daily Heliocentric Places of Venus, computed first at eight days,
" " " Mars " " twelve days,
and the Moon's Hourly Places, computed first at twelve hours.

Another example is the production of the Sun's Co-ordinates for noon and midnight from the original computations for noon only. In this case also it suffices to predetermine the last digit, and the last digit only, of the midnight quantities and entrust the completion to the machine.

In some cases (for instance, the Heliocentric Places of Uranus and Neptune) the quantities may be very nearly in arithmetical progression, that is, the First Differences may be very nearly constant. It is therefore desirable that all the Keys, except the one for the last digit, should be depressed in one operation only, so as to obviate needless attention, repetition, nerve action, loss of time, and danger of error. This assistance is given by an accessory, by means of which a set of key-depressors act collectively instead of human fingers acting individually (see fig. 3).

An example of actual work done on this machine is shown in the illustration below, with accompanying explanation.

EXAMPLE OF INTERPOLATION COMPLETED BY A SPECIAL BURROUGHS MACHINE.

	Heliocentric Longitude of MARS.				Heliocentric Longitude of MARS.	
				(35 13,51)		
1923 Jan.13	(·31 21 51)	,31	(3,22)		31 21 51,31	1923 Jan.13
		9		35 10,29		
		0 4	3,24		31 57 01,60	14
		5		35 07,05		
		5 5	3,25		32 32 08,65	15
		0		35 03,80		
		5 6	3,26		33 07 12,45	16
		4		35 00,54		
		9 8	3,28		33 42 12,99	17
		6		34 57,26		
		5 8	3,28		34 17 10,25	18
		8		34 53,98		
		3 9	3,29		34 52 04,23	19
		9		34 50,69		
		2 1	3,31		35 26 54,92	20
		8		34 47,38		
		0 2	3,32		36 01 42,30	21
		6		34 44,06		
		6 3	3,33		36 36 26,36	22
		3		34 40,73		
		9 4	3,34		37 11 07,09	23
		9		34 37,39		
		8 4	3,34		37 45 44,48	24
		5		34 34,05		
1923 Jan.25	(38 20 18)	,53			38 20 18,53	1923 Jan.25

DATA. MACHINE WORK.

```
PROCESS :-      First stage:-          Second stage:-
                         35 13,51
                            3.22     31 21 51,31
                         35 10,29------35 10,29
                              etc    31 57 01,60
                                          etc.
```

Illustration showing a wrong over-print during the second stage:-

34 34,05

An interesting use of the machine which is made possible by the device of "splitting," is the summing of two or more groups of terms at the same time. In this way the synthesis of small anharmonic quantities may be rapidly performed in conjunction with Professor E. W. Brown's device (*Monthly Notices of the Royal Astronomical Society*, vol. lxxii., No. 6, April 1912). It is well to notice that a mistake can easily be located without the need for doing the work again, seeing that all items are *recorded*.

Explanation of the Above Example.

By work previous to the machine 31° 21′ 51″·31 and 38° 20′ 18″·53 have been calculated from Newcomb's Tables for January 13 and 25 respectively.

Also, the last digits of the interpolated place for the intervening days have been predetermined, viz. 0.5.5.9.5.3.2.0.6.9.8, by (fundamentally) the well-known methods of interpolation, modified, however, to take advantage of the capabilities of the machine.

From the last digits of the longitude the last digits of the first and second differences are written down.

The process being supposed already complete up to January 13, it is then easily seen that the " 4 " of the second difference for January 14 means −3″·24, and all the second differences could now be easily set down in full.

The machine then builds up the first differences from the second differences, and subsequently the longitude from the first differences.

The guarantees of accuracy are :

(i) That the longitude calculated for every twelfth day is reproduced, *e.g.* 38° 20′ 18″·53 previously calculated from Newcomb's Tables is obtained by adding the first difference 34′ 34″·05 to 37° 45′ 44″·48.

(ii) When the human brain is relied upon to use differences, it is apt occasionally to make mistakes of the following nature :—34′ 34″·05 being taken as the quantity generated from the second differences, 34′ 34″·15 may be used as the quantity generating the interpolated longitude : and no record of the mistake is preserved. If the machine-operator makes a mistake of this nature, the result is that a " 1 " is printed over a " 0," as illustrated at the bottom of the example. This should not fail to catch the eye of the operator—in fact, a glance shows that in all the first differences of the example, the over-printed quantity is identical with the quantity below.

(2) Special Exhibition of the Nautical Almanac Anti-Differencing Machine. By T. C. Hudson, B.A., of H.M. Nautical Almanac Office ; by the courtesy of P. H. Cowell, M.A., D.Sc., F.R.S., Superintendent of the Nautical Almanac.

III. Mathematical and Calculating Typewriters.

(1) The Hammond Typewriter Co., Ltd.

THE new Multiplex Hammond Typewriter will write in either of two languages at a time, or in two different styles of type in any one language by merely turning a button. It has 350 different sets of type distributed over thirty languages which may all be used on the same machine, owing to the unique *interchangeable feature of the machine.*

There is no loose type, with a character on each type bar, as in other writing machines. In the Hammond the type is cast all in one piece, as in a

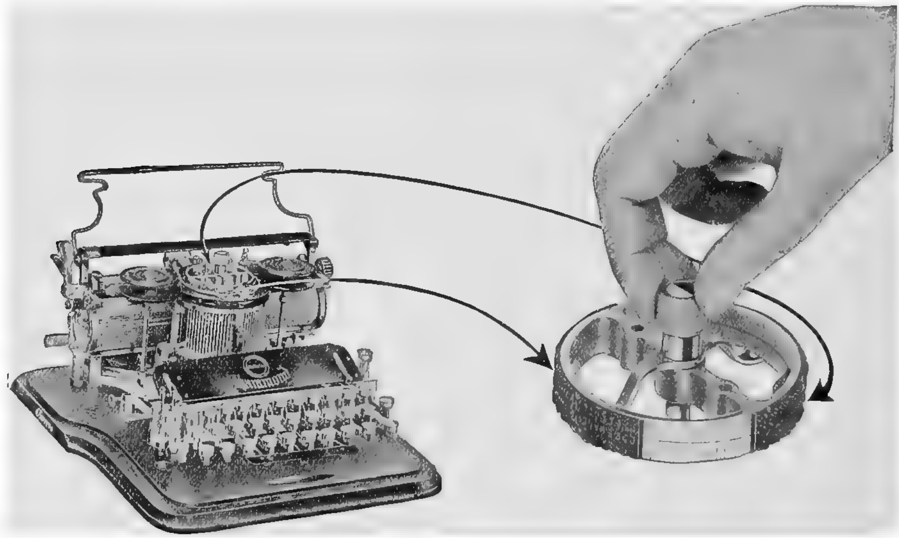


FIG. 1.

printing machine, and the operation of writing is performed upon a unique principle. Instead of type bars striking the paper through a ribbon, or by means of a pad, as in other machines, in the Hammond the paper is struck from *behind* with a constant blow, making every impression absolutely uniform, and giving any depth or intensity to the impression, according to the strength of the hammer blow, which can be varied by the operator at will.

This automatic action of the Hammond enables anyone who is not a typist to execute perfect work without any practice, because there is no touch *to learn, the impression being automatically uniform,* regardless of the operator's blow on the keys.

On one machine at one time there are always two different sets of type, each with either 90 or 120 different characters ; *instant change by the operator being possible*—even in the middle of a sentence.

The wide range of symbols provided makes it possible for the scientific man to write on the one machine almost any formula in mathematics, or to employ almost any language.

The Hammond Company show also a special mathematical model which will write any expression in the calculus and in higher mathematics generally, the same machine writing an ordinary letter in any language.

Greek, Turkish, Persian, Punjabi, Nagari, Arabic, Sanskrit, and many other Oriental languages are included. Where necessary the carriage operates in the reverse direction at the touch of a button.

It may be thought that such a versatile machine must necessarily be complicated, but, on the contrary, the Hammond claims to contain less than half the number of parts in any other standard typewriter. It is also portable.

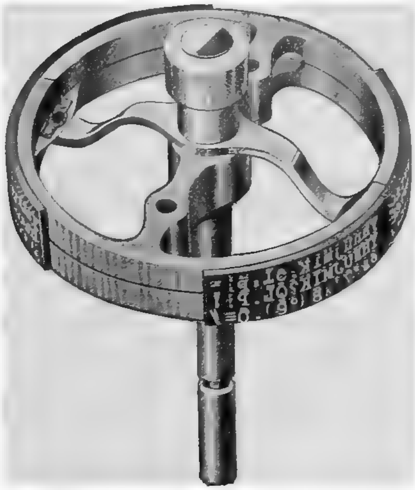


FIG. 2.

BARRETT ADDING MACHINE

The portable Barrett Adding and Computing Machine represents one of the most recent developments in calculating machines. It is simple in construction and claims to have 1100 parts less than the nearest competing machines.

No skilled operator is required, and the extreme portability of the Barrett enables it to be carried to the work.

It is made in over fifty different models, and in several styles, currencies, weights and measures.

EXHIBITS

1. One sterling, ten-column Barrett non-listing machine.
2. One decimal, ten-column, non-lister, with mezzanine keyboard.
3. One Mathematical Multiplex Hammond, containing *two complete sets of type, one for every expression* in higher or lower mathematics, and the other, one type out of 350 different styles in thirty languages.
4. One ordinary Multiplex Hammond, with universal keyboard, designed for scientific or professional use.

(2) The Monarch Wahl Adding and Subtracting Typewriter

This is an attachment to an ordinary correspondence typewriter, so arranged that the mechanism will add and subtract at will the figures placed in one or more columns as they are typed.

The actuator mechanism which lies in front of the machine is connected with the key levers which actuate the bars carrying the figures. The motion of these bars is communicated by the actuator to one universal gear wheel. When the key 1 is depressed, the universal gear wheel moves 1 tooth, and when the figure 9 is depressed, the universal gear wheel moves 9, and so on.

The other part of the mechanism is a totaliser which is carried on a truck immediately over the actuator, and is so arranged that the gears of the totaliser

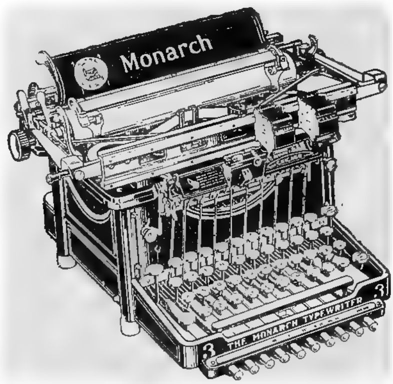


FIG. 3.

engage with the universal gear of the actuator. It will be seen that when the totaliser, which of course moves with the carriage, arrives at a position, say, for writing pounds, whatever amount is written will be recorded from the actuator to the totaliser.

The machine is fitted with a tabulating device which enables the operator, by a touch of the key, to place immediately the carriage carrying the totaliser in the correct position for typing predetermined amounts. For instance, if the operator wishes to write £342, 3s. 11d., he presses the tabulator key marked hundreds, and the carriage will then immediately travel to the correct position for writing the amount in question.

This tabulator works by means of stops which are carried in a magazine at the back of the tabulator. These stops, by one simple movement of the lever, are taken out of the magazine and deposited on the tabulator rack in any desired position. If it is desired to alter the setting of these stops, the "clear" lever will immediately take the stop off the rack and put it back in the magazine.

In the ordinary way the machine will of course add, but the mechanism can be reversed by a touch of the lever and the machine will then subtract.

There are numerous safeguards provided to prevent improper operation. If the operator starts to depress a figure key, the machine will automatically

lock until that key has finished its complete movement, and, in a similar way, that particular key cannot be depressed a second time until it has completed entirely its first movement. The totaliser, which is locked on the totaliser bar, can be removed, but immediately it is removed from the machine all its wheels are locked, and they cannot be moved until the totaliser is put back on to the machine.

When a two-colour ribbon is used, the colour of the writing, which changes automatically with each movement of the subtracting lever, shows whether the machine is adding or subtracting, and distinguishes clearly the subtractions on the page from the additions.

The machine is also provided with a disconnecting lever, the movement of which disconnects entirely the adding actuator from the figure keys, so that the machine becomes an ordinary typewriter.

The typewriter portion of the instrument is actuated by a nearly horizontal lever of the second species, called the key lever. At one end is the key, which is depressed by the operator, and at the other a fulcrum which is not fixed, but movable. Between the two is an attachment to a bell-crank which works the type-bar.

A feature of the machine is this change of position of the fulcrum. This is effected by making the upper edge of the fulcrum end of the lever slightly convex upward, and engaging with the lower side of a fixed plate, either horizontal or slightly convex downward. As the key is depressed the fulcrum moves from a position near the bell-crank attachment to a position far away. Thus there is an easy start, as the inertia of the moving parts is overcome rapidly, and the type-bar gives its stroke at its greatest speed, so that a sharp impression is formed.

Section E

THE ABACUS

The Calculating Machine of the East: the Abacus. Abridged from the Article on "The Abacus in its Historic and Scientific Aspects" in the *Transactions of the Asiatic Society of Japan* (vol. xiv., 1886). By Cargill G. Knott, D.Sc., F.R.S.E., Professor of Physics, Imperial University of Tokyo.

The Abacus possesses a high respectability, arising from its great age, its widespread distribution, and its peculiar influence in the evolution of our modern system of arithmetic. In the Western lands of to-day it is used only in infant schools, and is intended to initiate the infant mind into the first mysteries of numbers. The child, if he ever is taught by its means, soon passes from this bead-counting to the slate and slate pencil. He learns our Indian numerals, of which *one* only is at all suggestive of its meaning; and with these symbols he ever after makes all his calculations. In India and all over civilised Asia, however, the Abacus still holds its own; and in China and Japan the method of using it is peculiarly scientific. It seems pretty certain that its original home was India, whence it spread westward to Europe and eastward to China, assuming various forms, no doubt, but still remaining essentially the same instrument. Its decay in Europe can be traced to the gradual introduction and perfecting of the modern cipher system of notation, which again in part owes its early origin to the indications of the Abacus itself.

The *Soroban* or Japanese Abacus is one of the first objects that strongly attracts the attention of the foreigner in Japan. He buys at some shop a few trifling articles and sums up the total cost in his own mind. But the tradesman deigns not to perplex himself by a process of mental arithmetic, however simple. He seizes his Soroban, prepares it by a tilt and a rattling sweep of his hand, makes a few rapid, clicking adjustments, and names the price. There seems to be a tradition amongst foreigners that the Soroban is called into requisition more especially at times when the tradesman is meditating imposition; and in many cases it is certain that the Western mind, with its power of mental addition, regards the manipulator with a slight contempt. A little experience, however, should tend to transform this contempt into admiration. For it may be safely asserted that even in the

simplest of all arithmetical operations the Soroban possesses distinct advantages over the mental or figuring process. In a competition in simple addition between a " Lightning Calculator," an accurate and rapid accountant, and an ordinary Japanese small tradesman, the Japanese with his Soroban would easily carry off the palm.

Summary of Part I.: The Historic Aspect

The Abacus, as used in China and Japan, bears, on the very face of it, evidence of a foreign origin. The numbers are set down on it with the larger denomination to the left, a result which could come from a people either speaking and writing inversely, or speaking and writing directly. Historically, the home of the Abacus is in India ; but it could hardly have been invented by the Aryan Indians, who wrote directly and spoke inversely. The probability is they borrowed it from Semitic peoples, who were the traders of the ancient world ; and these may have invented it, or, as is perhaps more probable, received it from a direct-speaking, direct-writing race, such as we know the highly cultured Accadians to have been.

In early times the Abacus, as being an evolution from the natural Abacus—the human hand—pursued a course of development entirely different from that of the graphic representation of numbers. This latter we can trace through four stages,—the Pictorial, the Symbolic, the Decimal, and the Cipher. The Pictorial we find in the Egyptian hieroglyphics, the Accadian Cuneiform, and the technical Chinese of mathematical treatises ; the Symbolic in the numerous methods which grew up with the development of alphabets and syllabaries ; and the Decimal in the simplifications of these, which live to-day in the Chinese and Tamilic systems. Once the Decimal stage was reached, its general similarity to the Abacus indications suggested bringing them into still closer correspondence.

This advance seems to have taken place amongst the Aryan Indians, who, along with the Aryans of the West, very soon discarded the Abacus for the more convenient Cipher notation. With the Chinese, Tamils and Malayalams of South India, no advance was made in this direction ; the reason being simply that the Abacus better suited their numeration. These peoples speak directly, so that their nomenclature fits in perfectly with the Abacus indications, and makes its manipulation more rapid and certain than calculation by ciphering. An Aryan Indian with his inverse speaking could never work the Abacus with the same facility as a Japanese unless he worked from right to left—a mode of procedure quite foreign to his nature. It is not so foreign to Chinese and Japanese, however, to work from left to right, as each individual character is formed in this way. It may be safely concluded that only amongst a people who used the direct mode of naming numbers, or who with the inverse mode of naming preferred the inverse mode of manipulating, could the Abacus in the form in which it was evolved ever attain the beauty of action of the Japanese Soroban. To the discussion of its peculiar merits we now proceed. We shall employ throughout the Japanese name, which it should be noted is simply a mispronunciation of the Chinese name —*Swanpan*.

Part II.: THE SCIENTIFIC ASPECT

The Soroban may be defined as an arrangement of movable beads, which slip along fixed rods and indicate by their configuration some definite numerical quantity. Its most familiar form is as follows. A shallow rectangular box or framework is divided longitudinally by a narrow ridge into two compartments, of which one is roughly some three or four times larger than the other. Cylindrical rods placed at equal intervals apart pass through the ridge near its upper edge, and are fixed firmly into the bounding sides of the framework. On these rods the counters are "beaded." The size of the counters determines the interval between the rods, the number of which will of course vary with the length of the framework. Each counter (Japanese *tama*, or ball) is radially symmetrical with respect to its rod, on which it slides easily. Looked at from in front of the box, the form in perspective is that of a rhombus, the rod passing through the blunt angles. This double cone form makes manipulation rapid, the finger easily catching the ridge-like girth of the *tama*. On each rod there are six (sometimes seven) *tama*. Five of these slide on the longer segment of the rod, the remaining one (or two) on

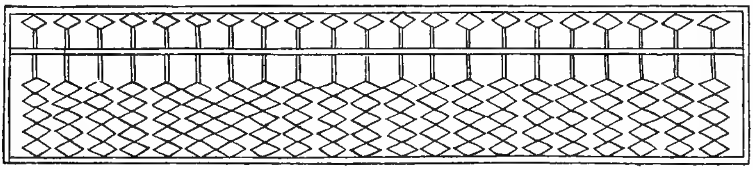


Fig. 1.

the shorter. When the *tama* on any segment of a rod are set in close contact, a part of the rod is left bare. The length of this bare portion is determined by a double consideration. It must be long enough to be clearly visible, and yet not so long as to make the action of the fingers irksome by reason of excessive stretching.

When a Soroban is lifted indiscriminately, the counters will take some irregular configuration upon their rods, being limited in their motions by the bounding walls and the dividing ridge. To prepare it for use, the framework is tilted slightly with the smaller compartment uppermost, so that each set of five counters slips down to the bounding wall end of its rod and each single counter[1] on its short rod slips down upon the upper surface of the dividing ridge. The framework is then gently adjusted till all the rods become horizontal, so that if any counter is shifted it will have no tendency to move back to its former position. By a sweep of the finger-tips along the surfaces of the single counters, these are driven from their contact with the dividing ridge to the other extremities of the rods. In this configuration, in which the counters are all as far away as possible from the dividing ridge, the Soroban is prepared for action. The number represented is zero. This position is shown in fig. 1.

[1] We shall henceforth only speak of *one* counter as being on the short rod. The two counters, although facilitating somewhat certain operations in division, are not really necessary, and their use is exceptional.

Let now any first counter of a set of five be moved till it is stopped by the ridge, as shown in the first diagram of fig. 2. This will represent 1, 10, 100, 1000, etc., as may be desired. Let it represent 1, then a second moved up will give us 2, a third 3, a fourth 4. This last is shown in the second diagram of fig. 2. The last moved up will of course give 5; but this number is also given by pushing back the five counters to their zero position and bringing down the corresponding single counter to the ridge. This is shown in the last diagram of fig. 2.

Leaving this single one in position, we get 6 by pushing up 1, 7 by pushing up 2, and so on till 9 is reached, as shown in fig. 3. The number 10 is then

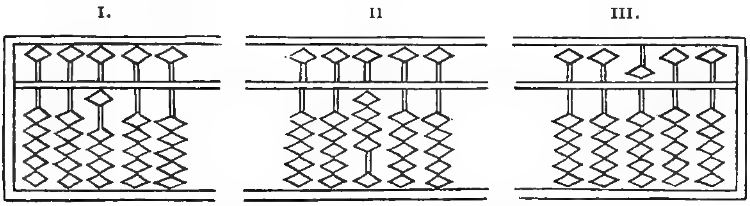


FIG. 2.

represented either by moving up the last counter, or more usually by clearing the rod of all its counters and moving one up on the next rod to the left, as shown also in fig. 3.

The mode of representing any number is thus obvious, being simply a mechanical model of our cipher system. Each rod corresponds to a definite figure "place" (Japanese *Kurai*) or power of ten. One being first chosen as the unit, the next to the left is the "tens," the next the "hundreds," the

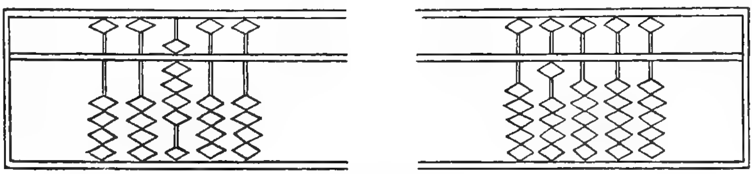


FIG. 3.

next the "thousands," and so on; while the successive rods to the right will represent the successive decimal places—tenths, hundredths, thousandths, etc. When the counters are as far as possible from the dividing ridge they have no value; when they are pushed as *near* the ridge as possible they have values as already indicated. The single counter when pushed down upon the ridge has five times the value of any other counter upon that rod. In fig. 4 the number 3085·274 is shown. The mark V is placed over the "units" rod.

The operations of addition and subtraction are self-evident. Thus, let it be required to add to this number 352·069. On the "hundreds" rod push up 3; and proceed throughout whenever it can be done in this way. On the "tens" rod, however, where only two counters are left, it is impossible to push up 5. But since 50 = 100 − 50, the addition is effected by pushing up one counter on the "hundreds" and removing 5 from the "tens" rod. This gives of course 4 on the "hundreds" rod and leaves 3 on the "tens."

Then push up 2 on the "units" rod; then 1 on the "tenths" rod with a simultaneous removal of 4 from the "hundredths" rod, since $10-6=4$; then 1 on the "hundredths" rod with a simultaneous removal of 1 from the "thousandths" rod. The final result 3437·343 is given in fig. 5.

Subtraction is executed in a similar manner. It will be noticed that these operations involve no mental labour beyond that of remembering the complementary number, that is, the number which with the given number makes up 10. A glance at the configuration on any rod is sufficient to show if the addition (or subtraction) of a named number can be effected on it; and if this cannot be, it is necessary simply to add (or subtract) one to (or from) the next higher place and subtract (or add) the complementary number from (or to) the place in question. In first experimenting with the Soroban, an operator who is accustomed only to our Western modes of figuring is apt to add mentally, and then set down the result on the instrument. Such a mode is inferior of course to the ordinary figuring method, being liable to error, inasmuch as the number that is being added is not visible to the eye at any time, and the number that it is being added to disappears in the operation.

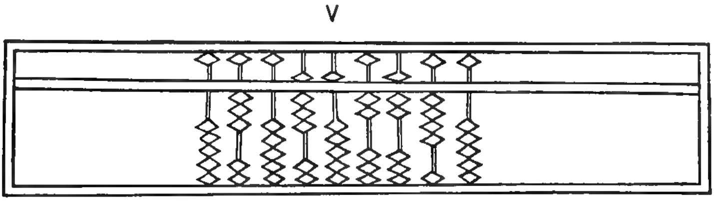


FIG. 4.

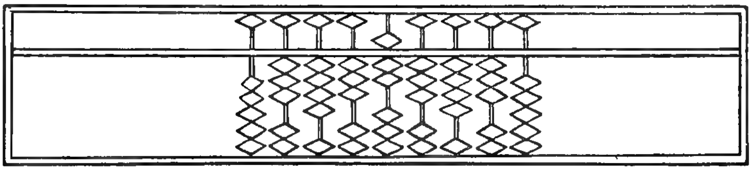


FIG. 5.

But if anyone will take the trouble to dispossess himself of his Western methods and work in the manner indicated, he will find Soroban addition and subtraction both more rapid and more certain, because attended by less mental exertion, than in figuring. The one seeming disadvantage in the Soroban is that the final result of each step alone appears, so that if any error is made, the whole operation must be carried through from the beginning again. Almost all writers on China or Japan, who have noticed the instrument, bring this forward as a serious disadvantage. But such a conclusion is a hasty one, and shows the writer to possess but small acquaintance with Soroban methods, and little regard to the true aim of calculation. For after all it is the result we wish; and if an error has been made, repetition is necessary both with Soroban and ciphering. The mean position of an accidental error is of course half-way through; and this would tell in favour of the ciphering system. But, on the other hand, the Soroban is, on the

average, much more rapid than ciphering, and less liable to error. Only a lengthened series of comparative experiments could establish whether there is any real disadvantage at all.

Multiplication

Multiplication on the Soroban differs but slightly from our own methods, being effected by means of a Multiplication Table—*ku ku gō sū*,[1] literally, nine-nine combining number. Two peculiarities distinguish this table from ours. First, there is a complete lack of interpolated words like our " times," the multiplier, multiplicand, and product being mentioned in unbroken succession ; and, second, the multiplier, that is the first-named number, is always the smaller. Thus the multiplication table for six runs :

ichi	roku	roku
ni	roku	jū ni
san	roku	jū hachi
shi	roku	ni jū shi
go	roku	san jū
roku	roku	san jū roku
roku	shichi	shi jū ni
roku	hachi	shi jū hachi
roku	ku	go jū shi

It is unnecessary to go to 12 as we do. Knowledge of a multiplication table for any number higher than 9 would retard Soroban manipulation.

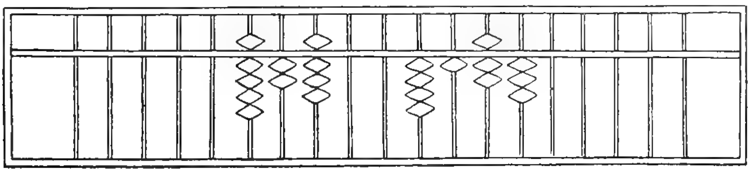


Fig. 6.

We British at least are compelled to learn up to 12 because of our monetary system ; and it is often serviceable to know the table for 16. One is early struck by the inability of most Japanese students to multiply by 12 or even 11 in one line.

In multiplying two numbers together on the Soroban, the operator sets the two numbers somewhat apart on the instrument, the multiplier being to the left, the multiplicand to the right. There must be left to the right of the multiplicand a sufficient number of empty rods, a number at least equal to the number of places in the multiplier. The operation is essentially the same as ours ; only, instead of multiplying the multiplicand by each figure of the multiplier as we do, the Japanese multiplies the multiplier by each figure of the multiplicand. As the operation goes on the multiplicand gradually disappears, so that finally only the multiplier and product are left on the board. An example will render the method clear. Let it be required to

[1] Generally called simply *ku ku*.

multiply 4173 by 928. Set these on the Soroban, the multiplier anywhere to the left, and 3 empty rods at least to the right of the multiplicand. Henceforward in the diagrams we shall represent visually only the counters which happen to be in use.

Multiply 8 by 3 and set 24 on the Soroban so that the 4 lies just as many

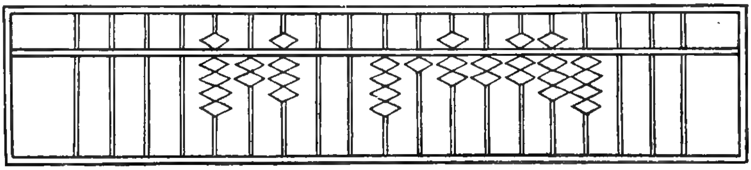


FIG. 7.

places to the right of the multiplicand 3 as there are figures in the multiplier. This 4 is of course in the " units " place of the product; and we shall continue to name the other places accordingly. Next multiply the 2 by 3, and add the product 6 to the " tens " rod. This gives us the result so far 84. Lastly, multiply 9 by 3. This requires 7 to be added to the " hundreds " rod, and 2 to the " thousands " rod. But before this latter operation can be done, the

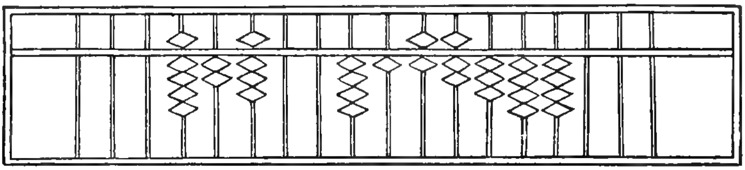


FIG. 8.

" thousands " rod must be cleared of its multiplicand 3, which having completely served its purpose may easily be removed, and indeed is better away. Since 3 is to be removed and 2 added, it is sufficient to remove 1 and leave 2. The result so far is shown in fig. 7.

Now proceed to multiply with the next figure of the multiplicand, 7, namely :—$7 \times 8 = 56$, of which the 5 is to be added to the " hundreds," and

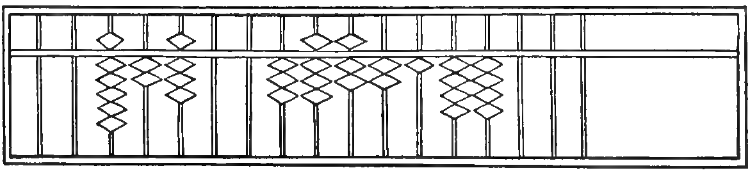


FIG. 9.

6 to the " tens " rod; $7 \times 2 = 14$, that is, 1 to the " thousands," 4 to the " hundreds "; $7 \times 9 = 63$, that is, leave 6 on the " ten thousands " rod by taking off 1 from the 7 and add 3 to the thousands. The result of this operation is given in fig. 8.

The operations with 1 and 4 are similarly carried out, care being taken to add the numbers which make up each several product in their proper places, and to suppress the multiplicand figure at the final operation with the same. The final result is given in fig. 9.

It will be noticed that in all addition or subtraction processes the number is added to or taken from the rod rather than from the number on the rod. The eye can tell at a glance if this operation can be effected on the rod in question, or if the next rod to the left has to be called into play. Mental labour is thus reduced to a minimum. The operator hears or utters a certain sound, which means one of two operations. A glance shows which of these it must be ; and the fingers execute a certain mechanical movement which accompanies the sound of the words as naturally as the fingers of a pianist obey the graphic commands of a Sonata.

We see then how well fitted for Soroban use is the Chinese and Japanese nomenclature of the numerals ; and how ill adapted all such systems must be which say sixteen and five-and-twenty or even sixteen and twenty-five instead of "teen-six" and twenty-five.

Division

Division on the Soroban, although essentially the same as our own Long Division, is in many respects peculiar and almost fascinating. The *art* of it is based upon a Division Table, called the *ku ki hō*, or Nine Returning Method, which is learned off by heart. This we give in full, with an accompanying translation as literal as possible.

Division Table for Ichi (*one*).

ichi is shin ga in jū	one one gives one ten
,, ni ,, ,, ni ,,	one two ,, two tens
,, san ,, ,, san ,,	,, three ,, three ,,

and so on to

ichi ku shin ga ku jū	one nine gives nine

Division Table for Ni (*two*).

ni ichi ten saku no go	two one replace by five
,, ni shin ga in jū	,, two gives one ten
,, shi ,, ,, ni jū	,, four ,, two tens
,, roku ,, ,, san jū	,, six ,, three ,,
,, has ,, ,, shi jū	,, eight ,, four ,,

This table could well stop at " ni ni shin ga in jū," since the higher ones are simply combinations of the first two. This is recognised by the absence of the " two five " statement.

Division Table for San (*three*).

san ichi san jū no ichi	three one thirty-one
,, ni roku ,, ,, ni	,, two sixty-two
,, san shin ga in jū	,, three gives one ten

The rest is obvious, being indeed but a repetition of the first three statements.

Division Table for Shi (*four*).

shi ichi ni jū no ni	four one twenty-two
,, ni ten saku no go	,, two replace by five
,, san shichi jū no ni	,, three seventy-two
,, shi shin ga in jū	,, four gives one ten

Division Table for Go (*five*).

go ichi ka no ichi	five one add one
,, ni ,, ,, ni	,, two ,, two
,, san ,, ,, san	,, three ,, three
,, shi ,, ,, shi	,, four ,, four
,, go shin ga in jū	,, five gives one ten

Division Table for Roku (*six*).

roku ichi ka ka no shi	six one below add four
,, ni san jū no ni	,, two thirty-two
,, san ten saku no go	,, three replace by five
,, shi roku jū no ni	,, four sixty-four
,, go hachi jū no ni	,, five eighty-two
,, roku shin ga in jū	,, six gives one ten

Division Table for Shichi (*seven*).

shichi ichi ka ka no san	seven one below add three
,, ni ,, ,, ,, roku	,, two ,, ,, six
,, san shi jū no ni	,, three forty-two
,, shi go jū no go	,, four fifty-five
,, go shichi jū no ichi	,, five seventy-one
,, roku hachi jū no shi	,, six eighty-four
,, shichi shin ga in jū	,, seven gives one ten

Division Table for Hachi (*eight*).

hachi ichi ka ka no ni	eight one below add two
,, ni ,, ,, ,, shi	,, two ,, ,, four
,, san ,, ,, ,, roku	,, three ,, ,, six
,, shi ten saku no go	,, four replace by five
,, go roku jū no ni	,, five sixty-two
,, roku shichi jū no shi	,, six seventy-four
,, shichi hachi jū no roku	,, seven eighty-six
,, hachi shin ga in jū	,, eight gives one ten

Division Table for Ku (*nine*).

ku ichi ka ka no ichi	nine one below add one
,, ni ,, ,, ,, ni	,, two ,, ,, two
,, san ,, ,, ,, san	,, three ,, ,, three

and so on to

ku hachi ka ka no hachi	nine eight below add eight
,, ku shin ga in jū	,, nine gives one ten

[In practice some of these phrases are contracted, such as *nitchin in jū* instead of *ni ni shin ga in jū*, *roku chin in jū* for *roku roku shin ga in jū*, and the like. The two words *ka ka* are run into one, *kakka*, the double *k* being strongly pronounced as in Italian. (Added, 1914.—C. G. K.)]

It will be noticed that the essential parts of the division tables take no account of the division of a number higher than the divisor. Hence in division, the larger number is named first; whereas in multiplication, as we saw above, the smaller number is named first. Thus the Japanese gets rid of such interpolated words as "times" and "into" or "out of," which are necessary parts of our multiplication and division methods.

In order clearly to understand this table, we must bear in mind that division is always at least a partial transformation from the denary scale to the scale of notation of which the divisor is the base. The adoption of the denary or decimal scale by all civilised notation is due entirely to the fact

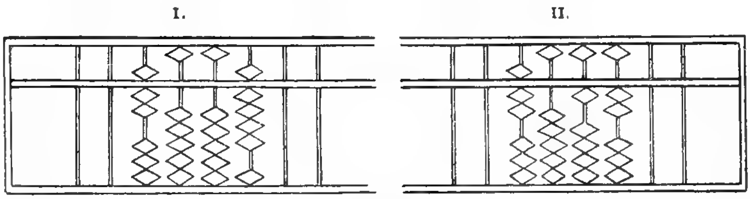


FIG. 10.

that man has ten fingers. There is no other peculiar charm about it; in some respects the duodenary scale would certainly be superior. As a simple example let us divide nine by seven; we get of course once and two over. This means that the magnitude which is represented by 9 in the denary scale is represented by 12 in the septenary scale. In this case the transformation is complete. We may test the accuracy of our work by writing down the successive numbers in the two scales.

Denary	1	2	3	4	5	6	7	8	9
Septenary	1	2	3	4	5	6	10	11	12

Now let us work out the problem on the Soroban. Set down the number 9 with 7 a little to the left. The division table for seven takes no account whatever of the number nine; but it says "shichi shichi shin ga in jū," or, as it might be paraphrased, "seven seven gives one ten"—where "ten" signifies not the number but the rod. As the operator repeats this formula, he removes 7 from the nine and pushes 1 up on the next rod to the left. The operation is shown in diagram 1 of fig. 10.

Now this number, represented by 12 in the septenary scale, we cannot call twelve, because twelve means ten and two, whereas here we have only seven and two. Practically we keep the unit as in the denary scale and use the phrase two-sevenths, which really signifies two in the septenary scale. A more complex example will make it clearer. Let it be required to divide 95 by 7; in other words, how many times is 7 contained in 95. By ordinary processes we obtain 13 and 4 over. This 4 is in the septenary scale; but 13 is still in the denary scale. Hence the transformation is only partial. To complete the transformation into the septenary scale we must express the denary 13 as the septenary 16; so that finally the denary 95 = septenary 164.

In this septenary number the 6 means 6 sevens, and 1 means 1 seven-sevens; precisely as in the denary number 9 means from its position 9 tens. Practically, of course, we keep the quotient in the denary scale and say 13 and 4-sevenths. Now perform this on the Soroban. First, as before, we remove 7 from the 9 and move 1 up on the next rod to the left. The Soroban now reads 125, as shown in diagram 2 of fig. 11.

We have now to divide 25 by 7. The Soroban manipulator, however, does not look so far ahead, but deals simply with the 20, or, what is the same thing, the 2 on the "tens" rod. His division table says "Shichi ni ka ka no roku," or, as we may paraphrase it, "Seven out of two, add six below," which implies that the 2 is to be left as it is and 6 added to the next rod, to the right. (This is precisely the equivalent of seven out of twenty, twice and six.) Now it is

I. II. III.

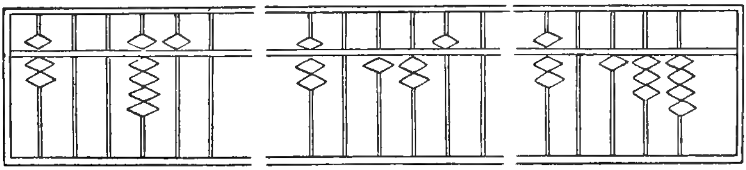


FIG. 11.

evident at a glance that we cannot add 6 to the next rod, which has already 5 on it. But, bearing in mind that we are still dividing by seven, we remove seven from the overfilled rod and push one up on the "tens" rod. Hence the operator is to add one to the "tens" rod, remove seven from, and add six to, the "units" rod; or simply add one to the "tens" rod and remove one from the "units" ($1=7-6$). The general rule is obvious. If the remainder number to be added to any rod equals or exceeds the number of unused counters on that rod, then one counter is pushed up on the rod immediately to the left, and from the first-named rod is subtracted that number which with the remainder makes up the divisor. Hence the final result stands as is shown in diagram 3 of fig. 11, where 4 appears as the remainder.

As another example let us divide 427,032 by 8. We may represent the operations symbolically thus, naming the successive results by *a*, *b*, *c*, *d*, *e*, *f*, and drawing a bar to show how far the operation has advanced. The translation of the Japanese verbal accompaniment to these operations is given below:

(8)	4	2	7	0	3	2
(*a*)	5 |	2	7	0	3	2
(*b*)	5	3 |	3	0	3	2
(*c*)	5	3	3 |	6	3	2
(*d*)	5	3	3	7 |	7	2
(*e*)	5	3	3	7	8 |	8
(*f*)	5	3	3	7	9 |	

(*a*) Eight four, replace by 5.

(*b*) Eight two, below add 4 (which being impossible means add 10[1] take off 4).

[1] This 10 is not "ten" but "eight," since for the moment we are working in the octenary scale.

(*c*) Eight three, below add 6.
(*d*) Eight six, seventy-four.
(*e*) Eight seven, eighty-six.
(*f*) Eight eight, gives one ten.

The chief advantage of the Soroban over ciphering lies in the absence of all mental labour such as is necessarily involved in the "carrying" of the remainder to the next digit. Once the Division Table is mastered and the fingers play obediently to the sound, the whole operation becomes perfectly mechanical. The only disadvantage is the often mentioned one, that the dividend disappears in the process. But this, as we have seen, is a small thing after all.

We shall now go through a problem in long division; and here the process is very similar to our own. Indeed, it can hardly escape notice that short division on the Soroban is essentially the same process as long division with us.

Let it be required to divide 703,314 by 738. Here again we shall symbolically represent the successive operations, so far as is necessary for clearness.

(738)		7	0	3	3	1	4
(*a*)	1	0	0	3	3	1	4
(*b*)		9	7	3	3	1	4
(*c*)		9	3	9	1	1	4
(*d*)		9	5	4	1	1	4
(*e*)		9	5	2	2	1	4
(*f*)		9	5	2	8	1	4
(*g*)		9	5	3	1	1	4
(*h*)		9	5	3	0	0	0

The start is made by consideration of the first figure on the left of the divisor.

(*a*) Seven seven, one ten. Take account now of the next figure in the divisor, multiply it by the 1 already obtained in the quotient and subtract the product from the second place in the dividend. Clearly this is impossible. Now observe that the first two figures of the line opposite *a*, namely 10, are really in the septenary scale.

(*b*) Hence take 1 from 10 (not ten but really seven) and add 7 to the next lower rod.

(*c*) Use 9 as multiplier now; subtract 9 times 30 or 270 from 733 and then 9 times 8 or 72 from the remainder. This completes the first operation, and is essentially the same as the first stage in the ordinary long division method.

(*d*) Start afresh as before with "seven three, forty two."

But 2 is greater than 1, the unused counter on the corresponding rod. Hence add one to 4 on the second rod and subtract 5 (7—2) from the third rod.

(*e*) Use 5 as multiplier; subtract 5 times 30 from 411, and 5 times 8 from the remainder.
(*f*) Start once again with "seven two, add six below."
(*g*) "Seven seven, gives one ten," which means—add one to the third rod, subtract seven from the fourth.
(*h*) Use 3 as multiplier; subtract 3 times 30 from 114, and 3 times 8 from the remainder.

Here again in the complete absence of any mental labour lies the peculiar merit of the Soroban. The only operation which calls for special remark is *a*, in which the first figure of the quotient is obtained by a process singularly rapid and free from all concentration of mind.

It is not necessary for rapid manipulation of the Soroban that one who is accustomed to Western modes of thought should use the Japanese Division Table. We may substitute our own peculiar method of dividing. There are, however, two of the Japanese tables which are singularly beautiful in their construction, the one for 5 and the one for 9. For example, let us divide 240,635 by 5. The table says "five two, add two," which is exactly the equivalent ultimately of our statement that "five into twenty give four." We may show the process symbolically thus :—

<table>
<tr><td>(5)</td><td>2</td><td>4</td><td>0</td><td>6</td><td>3</td><td>5</td></tr>
<tr><td></td><td>4 |</td><td>4</td><td>0</td><td>6</td><td>3</td><td>5</td></tr>
<tr><td></td><td>4</td><td>8</td><td>0 |</td><td>6</td><td>3</td><td>5</td></tr>
<tr><td></td><td>4</td><td>8</td><td>1</td><td>2 |</td><td>3</td><td>5</td></tr>
<tr><td></td><td>4</td><td>8</td><td>1</td><td>2</td><td>6 |</td><td>5</td></tr>
<tr><td></td><td>4</td><td>8</td><td>1</td><td>2</td><td>7 |</td><td></td></tr>
</table>

The process simply amounts to multiplying by 2 and dividing by 10; but with the Soroban it is peculiarly rapid.

Again let us divide the same number by 9. The table says "nine two add two below," which is identical in result with "nines in twenty twice and two," and so with the others. Symbolically we have :—

<table>
<tr><td>2</td><td>4</td><td>0</td><td>6</td><td>3</td><td>5</td></tr>
<tr><td>2 |</td><td>6</td><td>0</td><td>6</td><td>3</td><td>5</td></tr>
<tr><td>2</td><td>6 |</td><td>6</td><td>6</td><td>3</td><td>5</td></tr>
<tr><td>2</td><td>6</td><td>6 |</td><td></td><td></td><td></td></tr>
</table>

Here we cannot add 6 below; but instead we take off 3 (9−6) and put on one above as usual. Hence we obtain :—

<table>
<tr><td>2</td><td>6</td><td>7 |</td><td>3</td><td>3</td><td>5</td></tr>
<tr><td>2</td><td>6</td><td>7</td><td>3 |</td><td>6</td><td>5</td></tr>
<tr><td>2</td><td>6</td><td>7</td><td>3</td><td>7 |</td><td>2</td></tr>
</table>

The 2 is the remainder of course.

EXTRACTION OF SQUARE ROOT (*Kai hei hō*)

This requires, as in the ordinary ciphering process, a knowledge of the squares of the nine digits ; but its peculiarity lies in the use of another table of half-squares, *Han ku ku*. In both the Soroban and ciphering processes, the basis is the algebraic truth that the square of a binomial is the sum of the squares of the two components together with twice their product, or the corresponding geometrical theorem that if a straight line be divided into two parts, the square on the whole line is equal to the sum of the squares on the two parts together with twice the rectangle contained by the parts. In the arithmetical extraction of square root, the quantity is considered as consisting of two parts, the first part being that multiple of the highest power of 100 contained in the number which is a complete square. Thus the number 6889 is divided into 6400 and 489. But

$$6400+489=80^2+489$$

so that 80 is the first approximation to the value required. If we compare this with the binomial expression

$$\begin{aligned}(a+b)^2&=a^2+2ab+b^2\\&=a^2+(2a+b)b\end{aligned}$$

we see that our next operation must be to form the divisor $2a+b$, that is, in the numerical case $160+$ a quantity still unknown, but this quantity still unknown is also the quotient of the remainder 489 by the divisor. The process is to use 160 as a trial divisor, so as to get an idea what the unknown quantity may be. In this case we obtain 3, which added to 160 gives 163 ; and this multiplied by 3 gives 489. Hence the square root of 6889 is 83. Now in this mode of procedure a divisor quite distinct from the final result has to be formed. In the Soroban, however, whose peculiar feature in all operations is the disappearance of the various successive operations as the result is evolved, a distinct divisor does not appear. Thus, by an obvious transformation, we have

$$(a+b)^2=a^2+2\left(a+\frac{b}{2}\right)b.$$

Comparing this as before with

$$6889^2=80^2+489$$

we see, that by *halving* the remainder 489, we may employ a itself, that is 80, as our trial divisor. In completing this step we must take $\frac{1}{2}b^2$ instead of b^2 ; and hence the importance in the Soroban method of the table of half squares. The simplicity of the method will be recognised from the following example. It is required to extract the square root of 418,609. As in ordinary ciphering, tick off the number in pairs, beginning at the right hand. Then clearly 600 is the first approximation to the value of the square root, or 6 is the first figure in the answer. Move up 6 on a convenient rod somewhat to the left. The successive operations are given symbolically below, the description following as in the previous examples.

	6	4	1	8	6	0	9
(*a*)			5	8	6	0	9
(*b*)			2	9	3	0	4·5
(*c*)	64			5	3	0	4·5
(*d*)				4	5	0	4·5
(*e*)					3	0	4·5
(*f*)						2	4·5
(*g*)	647						0

(*a*) Subtract 6^2 or 36 from 41, leaving 5.
(*b*) Halve the whole remainder 58,609.
(*c*) Use 6 as trial divisor of 29. This gives 4. Subtract 4×6 or 24 from 29, leaving 5, and consider 64 as the full divisor.
(*d*) Subtract half the square of 4 from 53. This completes the second stage.
(*e*) Start with 6 again as trial divisor of 45, or more accurately 600 as trial divisor of 4504·5. This gives 7. Subtract 7×6 or 42 from 45.
(*f*) Subtract 7 times 40 from the remainder 304·5.
(*g*) Subtract half the square of 7 from the remainder 24·5. 647 thus appears as the last divisor and, as there is no remainder, it is the square root of 418,609.

The whole process may be easily proved by considering the expansion of the square of a polynomial. Take, for example, the quadrinomial $(a+b+c+d)$

$$\begin{aligned}(a+b+c+d)^2 = a^2 &+ b^2 \quad + c^2 \quad + d^2\\ &+ 2ab + 2bc + 2cd\\ &+ 2ac + 2bd\\ &\qquad\; + 2ad\end{aligned}$$

$$\begin{aligned}= a^2 + 2\Big[\Big(a + \frac{b}{2}\Big)b\\ + \Big(a + b + \frac{c}{2}\Big)c\\ + \Big(a + b + c + \frac{d}{2}\Big)d\Big]\end{aligned}$$

Extraction of Cube Root (*Kai ryu hō*)

The difference in the Soroban and ciphering processes arises from the same cause as in the case of square root. That is, instead of preparing a divisor, the Soroban worker prepares the dividend. The much greater complication in the case of the cube root necessitates an *undoing* of the processes of preparation at each successive stage—a mode of operation which was obviated in the case of square root by the use of the table of half-squares. The analogous table of "third cubes" would be excessively awkward in operating with, because of the decimal non-finiteness of the fractions of three. The operator is expected to know by heart the table of cubes, or *Sai jō ku ku*.

As in the ordinary ciphering method, the Soroban method depends upon the expression for the cube of a binomial. Consider, for example, the number 12,167. The first operation is to tick off in *threes*, that is in groups of ten-cubed. Now 12 lies between the cubes of 2 and 3. Hence 20 is the first approximation to the cube root of 12,167. We have

$$\begin{aligned} 12{,}167 &= 8000+4167 \\ &= 20^3 +4167 \end{aligned}$$

Now comparing this with the expression

$$\begin{aligned} (a+b)^3 &= a^3+3a^2b+3ab^2+b^3 \\ &= a^3+(3a^2+3ab+b^2)b \end{aligned}$$

we see that we must form a divisor whose most important part is $3a^2$, that is, 3×400 or 1200. Using 1200 as trial divisor of 4167, we get 3, which corresponds to the b in the general expression. We now form the complete divisor by adding to 1200 the expression

$$\begin{aligned} 3ab+b^2 &= 3\times 20\times 3+3\times 3 \\ &= 180+9 \\ &= 189 \end{aligned}$$

Thus we find as final divisor 1389, which, multiplied by 3, gives 4167; and hence 23 is the answer required.

The method on the Soroban depends upon the following transformation of the binomial expression

$$(a+b)^3=a^3+3a\left(a+b+\frac{b^2}{3a}\right)b$$

Here, by dividing the remainder (after subtracting the cube of the first member) by that member and by 3, we obtain an expression whose principal part is ab, that is, the product of the first member and the as yet unknown second member. Hence, using a as trial divisor of the first figures of the prepared dividend we get b. In the process the a or first member of the answer is set down in such a position relatively to the original expression that the b when it is finally evolved falls into its proper place succeeding a. We now subtract b^2 from its proper place in the remainder; and the final remainder obtained is $b^3/3a$. Operating upon this by multiplying first by 3 and then by a, that is, by an exact reversal of the original process of preparation, we get b^3 left. We shall illustrate the process by extracting the root of 12,167 according to the Soroban method. The number is first ticked off by threes in the usual way, and the first member of the answer is set down on the first rod to the left of the highest triplet. In this particular example there are only two significant figures in the highest triplet, so that the 2 is set down two rods to the left of the first figure in the original number. The successive steps are as follows; and as position is of supreme importance in this operation, we shall symbolise the Soroban rods by ruled columns :—

(*a*)	2		1	2	1	6	7
(*b*)	2			4	1	6	7
(*c*)	2		2	0	8	3	1
(*d*)	2		6	9	4	3	2
(*e*)	2	3	0	9	4	3	2
(*f*)	2	3			4	3	2
(*g*)	2	3			1	3	1
(*h*)	2	3				2	7
(*i*)	2	3				0	0

(*a*) Tick off into powers of 10^3 and consider the significant figures in the highest triplet, in this case 12. Two rods to the left set down 2, the highest integer whose cube (8) is less than 12.

(*b*) Subtract 2^3 or 8 from 12; or, to be more precise, subtract 20^3 or 8000 from the original number.

(*c*) Divide the remainder by the 2, which is the first found member of the answer. This, in accordance with the Soroban method of division, requires the first figure of the quotient to be set down one rod to the left. Also it must be noted that the last *unit* is a fractional remainder and means really one-half.

(*d*) Divide by 3, carrying out the process until the last rod with the $\frac{1}{2}$ remainder is reached. To this unit the unit of the fraction one-third which appears as a final remainder is added; so that the 2 on the last rod really means one-half and one-third. The division by 3 might be stopped at the preceding rod, so that instead of 69,432 we should have 69,411, in which the first unit means $\frac{1}{3}$ and the second $\frac{1}{2}$. There is greater chance of confusion, however, in this method than in the one shown, as will be seen when we come to the later stages.

(*e*) Divide by 2, but stop when the first figure in the quotient, in this case 3, is obtained.

(*f*) Continue this operation of division, regarding the newly obtained 3 as part of the divisor; or, in other words, subtract 3^2 or 9 from the next place to the right. We have now left a remainder represented by 43 and $\frac{1}{2}$ and $\frac{1}{3}$. This remainder is of the form $\frac{b^3}{a3}$; and to bring it back to a workable form we must multiply it by $3a$. We must be careful, however, to do this so as to take proper account of the peculiar mixed fraction represented by 2 on the last rod to the right. The next two stages effect this.

(*g*) Multiply by 3, beginning, however, at the second last rod, and thus undoing the operation *d*. Multiplication on the Soroban is accompanied by displacement to the right. Hence the product 3×43 or 129 has its last right-hand figure added to the rod containing the mixed remainder 2; and the final result of this operation gives 131, in which the last unit means as before one-half.

(*h*) Multiply by 2, beginning with the second last rod, and thus undoing the effect of operation *c*. The product 2×13 or 26 is added to the 1, and the 27 appears as the final expression.

(*i*) Subtract 3^3 or 27, and the remainder is zero.

Had we stopped in the operation *d* at an earlier point as suggested, we should have had to modify the reverse operation *g*. Thus, only the 4 of 411 would need to be multiplied by 3, giving of course 12 to be added to the first of the two units. The final result would have been of course 131, as already obtained.

The processes for extracting square root and cube root, on the other hand, imply a knowledge of mathematics much wider than the Abacus itself could ever teach. Square Root might perhaps have been evolved as a purely arithmetical operation on the Abacus; but Cube Root certainly could not. It seems more reasonable to suppose that both processes were deduced by some more general mathematical method, either algebraic or geometric.

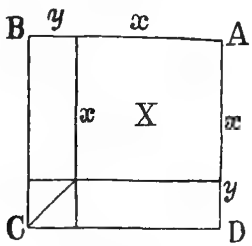


Fig. 12.

The geometrical aspect is indeed most instructive. Consider, for example, the square A B C D, from which has been subtracted the small square X, whose side x is known in finite terms. The L-shaped portion measures the remainder after X has been subtracted from the large square. From this remainder we have to find the length y, which with x makes up the side of the large square. The line drawn from C to the contiguous corner of X evidently cuts the L-shaped remainder into two halves. And each half is made up of the product of x and y and half the square of y. Here we have at once the suggestion of the Abacus rule for extracting square root. A similar consideration of the properties of the cube would lead to the Abacus rule for extracting the cube root. It is not probable, however, that these rules were discovered in this way. They are rather to be regarded as having been deduced from general algebraic considerations, just as our own rules are. They involve a knowledge of the binomial theorem, not necessarily in its complete generality, but so far at least as positive integers are concerned. It is known, however, that Chinese mathematicians have been acquainted for centuries with the binomial theorem, which they employed in the solution of equation of high degree. Hence it is almost certain that the Abacus rule for cube root is a formula deduced from the algebraic mode of solving such an equation as

$$x^3 - a = 0$$

The rule, of course, had to be formulated so as to suit the peculiar conditions of the arithmetic Abacus. The discussion of what might be called the algebraic Abacus or chess-board-like arrangement for solving equations is beyond the scope of the present paper.

See in this connection *A History of Japanese Mathematics*, by David Eugene Smith and Yoshio Mikami (Chicago, 1914).

Exhibits

1. Japanese Abacus. Lent by Cargill G. Knott, D.Sc.
2. Chinese Abacus. Lent by Major W. F. Harvey, I.M.S.

PORTRAIT OF JOHN NAPIER.

From a Drawing in the possession of the Earl of Buchan.

[*To face p.* 30.

SECTION F

SLIDE RULES

The Slide Rule. By G. D. C. STOKES, D.Sc.

(1) A SUMMARY OF THE HISTORICAL DEVELOPMENT OF THE SLIDE RULE TO 1850

(The references are to F. Cajori's *History of the Logarithmic Slide Rule* and to the *Mechanics' Magazine*, 1831, vol. xiv.)

1620. Gunter invented the straight logarithmic scale and effected calculation with it by the aid of compasses. It was subsequently used in navigation. (p. 1.)

1628. Wingate used a fixed scale giving logarithms and antilogarithms. (Disputed, pp. 5–10 and Addenda.)

1630. Oughtred invented the straight logarithmic slide rule. His instrument consisted of two rulers slid along each other and kept together by hand. He also invented the circular Gunter scale. Published 1632. (p. 11.)

1630. Delamain constructed the first circular slide rule. (Disputed, p. 14, and *Mech. Mag.*, pp. 5, 6.)

1650. Milburne designed the first spiral logarithmic scale. (p. 15.)

1654. Rules in which the slide worked between parts of a fixed stock were known in England (see p. 163 of this volume). Formerly this invention was credited to Partridge (1657). (p. 17.)

1675. Newton solved the cubic equation by means of three parallel logarithmic scales, and made the first suggestion towards the use of a runner. (p. 32.)

1722. Warner used square and cube scales. (p. 27.)

1755. Everard inverted the logarithmic scale, and adapted the slide rule to gauging. (p. 18.)

1755. Leadbetter used three slides on one rule. (p. 29.)

1768. The use of the inverted slide was known in England. This inversion was proposed subsequently by Pearson (about 1797). (*Mech. Mag.*, p. 5.)

1775. Robertson constructed the first runner. (p. 32.)

1787. Nicholson designed the logarithmic scale in sections, and displaced fixed scales relatively. He also used the slide in the manner of the Gunter compasses. (p. 35.)

1815. Roget invented the log-log scale. (p. 38.)

1840. Woolgar generalised the logarithmic scale and applied the slide rule to annuities. (p. 50, and *Mech. Mag.*, p. 308.)

1842 ? Macfarlane used a slide rule having scales of equal parts with numbers in geometric progression. (p. 50.)

1850. Mannheim designed the modern standard British slide rule, constructed the first cylindrical type, and popularised the runner. (p. 63.)

The subsequent development has been mainly along the lines of (1) extension of the length of the scales without a corresponding increase in the size of the instrument ; (2) adaptation to specialised branches of science ; and (3) increase of mechanical efficiency. Among names associated with (1) may be mentioned Everett, Hannyngton, Thacher, Fuller, Barnard, R. H. Smith, Anderson, and Proell ; and among a still greater number in (2), Baines, Hudson, Fürle, Smith-Davis, Maitland, and Strachey.

(2) Classification of Slide Rules

The term " slide rule " has never been restricted to rules in which sliding was an essential feature. There is thus a class of rules for which the name " logarithmic computing scales " would be more appropriate. M. d'Ocagne classifies slide rules conveniently under two heads : (1) rules worked by movable indices ; (2) rules with adjacent sliding scales. There is also an intermediate type in which sliding takes place without performing the function of displacing the scales relatively to one another. Examples of class (1) are the circular scales of Oughtred (1630), Scott (1733), Nicholson (1787), Weiss (1901) ; and the spiral scales of Milburne (1650), Adams (1748), Nicholson (1798), and Lilly (1912). In principle these rules are Gunter scales : in multiplying by them log a is measured by some form of dividers and added to log b by applying one arm of the dividers to point b on the scale.

Among the intermediate class are the straight rule of Nicholson (1787), the circular calculator of Boucher (1876), and the modern helical forms of Fuller, Barnard, and Smith. In Nicholson's rule the slide carried no scale, but took the place of the dividers. In the Boucher instrument one dial moves relatively to the other : nevertheless multiplication and division are performed by the Gunter method. In the helical rules one index is fixed and the scale made movable, but the mode of operating is again that of Gunter.

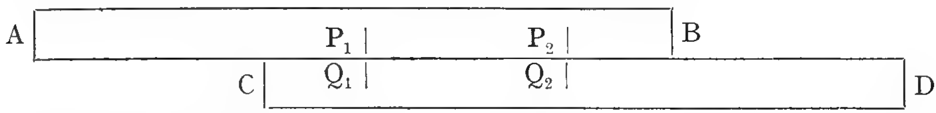
The number of rules coming under class (2) is very great. Among earlier ones may be noted the straight rules of Partridge (1657), Everard (1755), Roget (1815), Mannheim (1850) ; the circular forms of Biler (1696), Clairaut (1727), Sonne (1864), Charpentier (1903) ; and the cylindrical design of Thacher (1881). Present-day designs are given under the special descriptions.

(3) Mathematical Principle of the Slide Rule [1]

By common practice the term slide rule is used in the sense *logarithmic* slide rule, and thus slide rules are generally regarded as a direct development of the work of Napier. Historically and practically this is true. It is possible, however, to have slide rules independent of logarithms.

[1] See Runge, *Graphical Methods*, pp. 43–52, 1912, Columbia Univ. Press.

Let us consider two ways of tabulating in the form of a scale the values of a single-valued function $f(x)$ corresponding to any range of values of its argument x. In the first way an equi-interval series of values of x may be taken and represented by equal intervals on a scale, and the calculated values of the function marked down on the points of division. If such a scale AB be constructed for $f(x)$, and a scale CD for $g(t)$ with the same size of divisions, and the two scales be set alongside, the relation between readings at two pairs of corresponding points P_1, P_2 on AB and Q_1, Q_2 on CD is determined by $x_2-x_1=t_2-t_1$. If only the *functional* values and not values of the argument are marked on each scale, and we take p_1, p_2 to denote readings



at P_1, P_2, and q_1, q_2 for readings at Q_1, Q_2, then $p_1=f(x_1)$, $q_1=g(t_1)$, etc., and we get

$$f^{-1}(p_2)-f^{-1}(p_1)=g^{-1}(q_2)-g^{-1}(q_1) \qquad . \qquad . \qquad (1)$$

where $f^{-1}(x)$ means the function inverse to $f(x)$. Such a system of sliding scales therefore solves equation (1) for any one of the quantities p_1, p_2, q_1, q_2 in terms of the other three. An example of this rule is that of Macfarlane (1842), who took $f(x)=a^x=g(x)$. Now $p=a^t$ gives the inverse relation $x=\log_a p$; hence (1) becomes $p_2/p_1=q_2/q_1$, the fundamental property of the logarithmic slide rule. More recently this method has been discussed by J. A. Robertson (*Journal of the Inst. of Actuaries*, vol. xxxii. p. 160), who used a table of antilogarithms on the principle of the Gunter scale.

In the second method of scalar tabulation the quantities marked are values of the *argument* (conveniently at equal intervals) at points whose distances from the origin of the scale are the respective functional values. Thus $AP_1=f(x_1)$, $CQ_1=g(t_1)$, so that

$$f(x_2)-f(x_1)=g(t_2)-g(t_1) \quad . \qquad . \qquad (2)$$

and this arrangement solves equation (2) for any one of x_1, x_2, t_1, t_2 in terms of the other three. When $f(x)=\log x=g(x)$, the proportion $x_2/x_1=t_2/t_1$ is again obtained; in both cases multiplication and division are performed mechanically. Equations (1) and (2) are of the same type fundamentally as the terms function and argument are purely relative. The single-slide slide rule may therefore be regarded generally as *an instrument for effecting mechanically the computation of one quantity in terms of other three when the four quantities are connected by the form stated in* (2).

But few indeed are the formulæ that come directly under this equation. Runge gives the case $f(x)=1/x=g(x)$, which solves $1/R=1/R_1+1/R_2$, since it can be put in the form $1/R-1/R_1=1/R_2-1/\infty$. Similarly, $f(x)=x^2$ and $g(t)=k\cos t$ effects the solution of $V^2-v^2=k(\cos a-\cos x)$. It is because the logarithmic function enables *products* and *powers* to be reduced to the difference form (2) that logarithmic forms of slide rule have outrivalled all others.

This reduction in the case of involution is effected by taking logarithms twice. Thus if $y=ax^n$, we get

$$\log y-\log a=n\log x-\log 1.$$

A rule graduated to $f(x)=\log x$ and $g(t)=n\log t$ would give y in terms of x and a, but only for the *one* value of n. Taking logarithms again,

$$\log\log y/a-\log\log x=\log n-\log 1.$$

Hence if the scales are graduated to log log x and log t, y/a can be read for any values of x and n, or n for any values of x and y/a.

It should be noted that the involution problem is also solved by the slide rule having equal divisions, if the scales are marked with the values of a^{a^x} and a^t respectively. In graduation the "exponential" slide rule is simple compared with the logarithmic slide rule, but its use is seriously limited by the practical difficulties of reading and setting, and it does not appear to have been exploited commercially.

(4) The Standard British Slide Rule

The Mannheim design is as follows:—On the face of the rule there are four scales, A, B, C, D, two of which (A, D) are on the stock, B and C being on the slide. The graduations are made so that on A or B the point marked x is distant log x units from the left end of the scale, and on C or D the point marked x is distant 2 log x units from the left end. Scales A, B thus range from 1 to 100; scales C, D from 1 to 10, the length of each scale being 2 units (usually 25 cm.). The back of the slide carries two scales (S and T), measuring 2 log (10 sin x) and 2 log (10 tan x) units respectively; and values of sines and tangents between 1 and 0·1 are thus read on the C scale against the right or left index. A joint scale for small angles $\sin^{-1}0{\cdot}1$ or $\tan^{-1}0{\cdot}1$ to $\sin^{-1}0{\cdot}01$ or $\tan^{-1}0{\cdot}01$ is given between the S and T scales. An ordinary scale of equal parts is usually given on at least one side of the stock.

Design of the log-log Scale (E)

This scale measures log log x between the points reading x and 10, which latter point is the zero of the scale. If a, b denote readings on E opposite 100 and 1 on A, we have

$$\log\log a-\log\log b=\log 100-\log 1,$$

whence $a=b^{100}$. Putting $a=10^n$, we get $b=10^{0{\cdot}01n}$. To locate the zero of the scale let k denote the A reading opposite 10 on E. Then

$$\log\log a-\log\log 10=\log 100-\log k,$$

whence $k=100/\log a=100/n$. Again, let an F scale be introduced giving reciprocals of numbers on E, so that a^{-n} may be calculated by finding a^n on E and reading the reciprocal on F. We then find the following ranges associated.

k.	*Range of* E *Scale.*	*Range of* F *Scale.*
33·3̇	1000 to 1·071	0·932 to 0·001
25	10000 ,, 1·096	0·912 ,, 0·0001
20	100000 ,, 1·122	0·890 ,, 0·00001

There is a gap between the E scale and the F scale near 1, and a number like $1{\cdot}05^{2\cdot7}$ could not be found by the slide rule for any of these designs. This gap, however, is filled by the binomial approximation

$$(1+x)^n = 1+nx+\tfrac{1}{2}n(n-1)x^2,$$

and in many cases the last term will not be required.

The foregoing applies to log-log scales designed to give powers and roots in conjunction with the A scale. In the Yokota rule the log-log scale is split into three sections and used in conjunction with the C scale, thereby increasing the accuracy. The following are among rules carrying a log-log scale: Blanc, Davis, Electro, Faber, Jackson-Davis (on a separate slide), Perry, Yokota.

(5) Functions read on the Standard Rule

There are four general ways of setting the slide by means of scales A, B, C, D. If a, b, c, d denote numbers on A, B, C, D respectively, these settings may be indicated by (a, b), (a, c), (b, d), (c, d), where (a, b) means a on A set opposite b on B. After the slide is set the runner may be set on any of the four scales and a reading can then be taken from the runner on two of the three remaining scales (for the scale which is fixed relatively to the one on which the runner is set would give readings independent of the setting of the slide). Instead of a reading by the runner the index readings on the S or T scales may be made. Conversely an index setting on S or T may be associated with eight ways of setting and reading the runner, namely, (a, b), (b, a), (a, c), (c, a), (b, d), (d, b), (c, d), (d, c). But on analysis the number of distinct forms obtained, though large, will appear to be much fewer than the number of operations for obtaining them.

Let a, b, c, d refer solely to setting the slide, and a', b', c', d' to setting or reading the runner after the slide has been set. Then each of the following equations expresses one distinct calculable form, and the notation also gives the rule for setting. The number in brackets gives the number of ways in which the expression may be calculated.

Forms calculable on Scales A, B, C, D

$$c' = \frac{cd'}{d} \quad (4)$$

$$x' = \frac{ac'^2}{b} \quad (4) \qquad c' = d'\sqrt{\frac{b}{a}} \quad (4) \qquad a' = \frac{ac'^2}{c^2} \quad (4)$$

$$c' = \frac{d'}{d}\sqrt{b} \quad (4) \qquad c' = \sqrt{\frac{a'b}{a}} \quad (2) \qquad a' = \frac{ab'}{c^2} \quad (2)$$

$$b' = \frac{c^2d'^2}{a} \quad (2) \qquad c' = \frac{1}{d}\sqrt{a'b} \quad (2) \qquad a' = \frac{d^2c'^2}{c^2} \quad (2)$$

Rule for setting.—Set opposite each other on their respective scales the numbers given by the two undashed letters; move the runner to the number (and on the scale) fixed by the dashed letter, and read the result at the runner on the scale given by the left-hand member of the formula.

Take, for example, the four ways of setting for $27\cdot3\sqrt{\frac{8\cdot42}{19\cdot15}}$, one of which is indicated above by the formula $c'=d'\sqrt{\frac{b}{a}}$. Set 8·42 on B opposite 19·15 on A, move the runner to 27·3 on D, and read the result on C (18·1). Second way: Set 8·42 on A opposite 19·15 on B, and read D opposite 27·3 on C. Third way: Set 19·15 on A opposite 27·3 on C, and read C opposite 8·42 on A. Fourth way: Set 19·15 on B opposite 27·3 on D, and read D opposite 8·42 on B.

With the sine and tangent scales in the ordinary position fewer operations are possible, as the runner cannot be set on them. For settings on the sine scale we find the forms

$$c'=d'\sin x,\qquad d'=c'\operatorname{cosec} x,\qquad b'=a'\sin^2 x,\qquad a'=b'\operatorname{cosec}^2 x,$$
$$a'=c'^2\operatorname{cosec}^2 x,\quad c'=\sqrt{a'}\sin x,\quad b'=d'^2\sin x,\qquad d'=\sqrt{b'}\operatorname{cosec} x.$$

For example, to calculate $\sqrt{22\cdot7}\sin 32°$ (formula $c'=\sqrt{a'}\sin x$), set S to 32°, move the runner to 22·7 on A, and read C (2·523).

Conversely, if the sine scale is read from settings on A, B, C, D, we can find the inverse sines of

$$\sqrt{\frac{b}{a}},\quad \frac{c}{\sqrt{a}}\cdot\quad \frac{\sqrt{b}}{d},\quad \frac{c}{d}.$$

If, however, the slide be turned over (scales S, T displacing B, C on the face of the rule), then the runner can be set at *any* points on S, T; hence the series of formulæ applicable to A, B, C, D hold good when 100 $\sin^2 s$ is substituted for b and 10 $\tan t$ for c. Again, the slide may be inverted directly, or turned over and then inverted. The effect in the former case is to substitute $100/b^2$ for b and $100/c$ for c; in the latter $\cot^2 x$ for b and $\operatorname{cosec}^2 x$ for c.

Without attempting to discuss the relations arising out of the log-log scale (E) in conjunction with A, B, C, D, S, T, we may note the simple cases

$$e'=e^{\frac{a'}{a}},\ a'=a\frac{\log e'}{\log e},\ k\log e=\operatorname{cosec}^2 x,\ a'=kc'^2\log e.$$

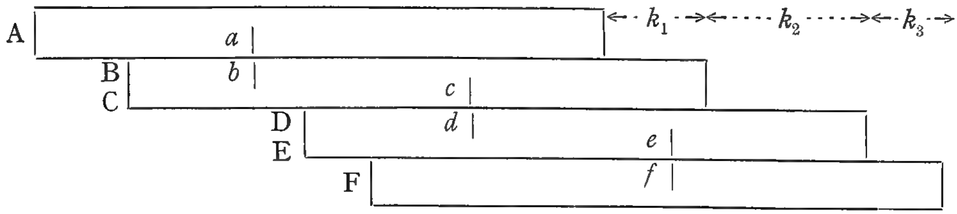
where k is the A reading opposite 10 on E and varies for different slide rules.

It is evident that the scope of the standard slide rule, even when limited to one setting of the slide plus one of the runner, is very great. Indeed, the practical computer does not attempt to learn any but the simplest operations, and meets more complex cases (when they do occur) by extending the number of simple settings.

(6) Poly-slide Rules

Consider the system of sliding scales in which the upper one has one scale A given by $f_1(x)$; the next slide two scales B, C given by $f_2(x)$, $f_3(x)$;

the third slide two scales D, E given by $f_4(x), f_5(x)$; and the last slide one scale given by $f_6(x)$.



Let the system be displaced by intervals k_1, k_2, k_3, as in the figure, and let a, b, etc., be pairs of corresponding readings on adjacent scales. Then

$$k_1 = f_1(a) - f_2(b)$$
$$k_2 = f_3(c) - f_4(d)$$
$$k_3 = f_5(e) - f_6(f).$$

Case 1.—*The Ordinary Two-slide Rule*

Let A and F be fixed on the stock while the two slides move independently. Then k_1 and k_2 are arbitrary, but $k_1 + k_2 + k_3 = 0$. Hence

$$f_1(a) + f_3(c) + f_5(e) = f_2(b) + f_4(d) + f_6(f)$$

with an obvious extension for the case of n slides. This two-slide rule enables the value of any one of a, b, c, d, e, f to be read when the values of the other five are known.

A good example of this type of design is furnished by the *Hudson Horse Power Computing Scale*. Let us deduce the formula from an examination of the scales. The A scale runs from 2500 to 25 : therefore $f_1(x) = \log 2500/x$. An index is fixed at distance log 12·5 along the B scale, so that $f_2(x)$ is constant and equal to log 12·5. Similarly, $f_3(x) = \log x/10$; $f_4(x) = \log 4{\cdot}2 + \log 10/x$; $f_5(x) = \log x$; $f_6(x) = 2 \log 100/x$. Hence

$$\log 2500/a + \log c/10 + \log e = \log 12{\cdot}5 + \log 42/d + 2 \log 100/f,$$

or

$$a = cdef^2/21000,$$

giving I.H.P. (a), in terms of revs. per minute (c), mean pressure (e), stroke (d), and cylinder diameter (f).

The foregoing example is a special case of the formula $F = xy^m z^n u^r v^s$, where m, n, r, s are constant. By taking logarithms this is easily reduced to the equation

$$\log F + \log 1/x + m \log 1/y = n \log z + r \log u + s \log v.$$

Hence one arrangement would be to graduate A to $\log x$, B to $n \log x$, C to $\log 1/x$, D to $r \log x$, E to $m \log 1/x$, and F to $s \log x$. This is possible if the numerical values of m, n, r, s are known and fixed.

An important limitation to the extension of involution to the poly-slide rule may be noted. Taking, for example, $F = x\, y^m z^n$ *with* m, n *variable as well as* x, y, z, F, it will be seen that taking logarithms twice does *not* effect a reduction to the form required for a two-slide design.

Reference may be made to a paper by J. W. Woolgar in the *Mechanics' Magazine*, 1831, vol. xiv. pp. 308-311, for a two-slide design applicable to annuities. The modern Essex Calculator also is an excellent example from hydraulics of the two-slide rule.

Case 2.—Slides with Dependent Motion

Let all the slides be connected by a mechanism which allows one setting between any two scales to be made, but which fixes all the other slides for that setting. If k_1 be assumed independent, $k_2=\phi(k_1)$ and $k_3=\psi(k_1)$, the functional forms being determined by the mechanism. Eliminating k_1, we find

$$f_3(c)-f_4(d)=\phi[f_1(a)-f_2(b)]$$
$$f_5(e)-f_6(f)=\psi[f_1(a)-f_2(b)]$$

simultaneously solved.

Only one rule of this type calls for notice, namely, the Baines slide rule. There is no scale-carrying stock in this rule, but four slides are connected by a parallelogram linkage, so that in every position $k_1=k_2=k_3$, giving

$$f_1(a)-f_2(b)=f_3(c)-f_4(d)=f_5(e)-f_6(f).$$

As the only advance these equations show upon those of the single-slide rule is that two special formulæ (not even wholly independent) can be dealt with instead of one, the advantage of applying the Baines design to Flamant's formula $V=76\cdot28\ d^{\frac{5}{7}}s^{\frac{4}{7}}$ is more apparent than real (see *The Engineer*, 1904, p. 346). But the Baines rule is noteworthy for introducing a dependent motion of the slides, an idea which may lead to future developments.

A Four-slide Rule

An example of four-slide design is furnished by the *Callender slide rule for determining the sizes of cables*. The chief mathematical interest in this instrument lies in the combination of four slides with a logarithmic chart. Two slides are horizontal and adjacent, the other two vertical and adjacent, and the result is read on the chart. Analysis of the arrangement leads to the formula $R=0\cdot513\ kfyW/V^2p$. There are thus seven variables, and the equation reduced to the general form given under case (1) becomes

$$\log R+2\log V+\log p+\log 1/0\cdot513=\log k+\log f+\log y+\log W.$$

Hence this case can be met by a three-slide design, which would be more compact and easy to read, though less easy to set.

Bibliography of the Slide Rule

F. Cajori, *A History of the Logarithmic Slide Rule*, 1909, London, Constable.

M. d'Ocagne, *Le Calcul Simplifié*, 1905, Paris, Gauthier-Villars.

C. N. Pickworth, *The Slide Rule: a Practical Manual*, 12th edition, 1910, London, Whittaker & Co.

Dunlop and Jackson, *Slide Rule Notes*, 1913, Longmans, Green & Co.

(A full bibliography is given in Professor Cajori's *History*)

Slide Rule Exhibits.

(1) Drawings of a Logarithmic Slide Rule made in the Year 1654.

Description and drawings by David Baxandall, A.R.C.Sc.

The instrument here represented is in the collection of mathematical instruments at the Science Museum, South Kensington, where it has been exhibited since 1898.

In a note in *Nature*, 5th March 1914, attention was called to the existence of this slide rule, and to its interest in connection with the early history of that instrument. As no other account has been published, the following detailed description is given here for purposes of reference :—

The instrument is of boxwood, well made, and bound together with brass at the two ends. It is inscribed : " Made by Robert Bissaker, 1654, for T. W." Up to the present no information about the maker has been found, and " T. W." remains unidentified. It is a little more than two feet in length, nearly an inch square in section, and bears the lines first described by Edmund Gunter. There are nineteen scales in all, as indicated

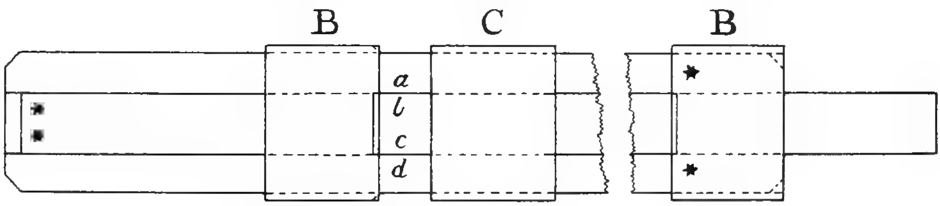


Fig. 1.

below. The divisions of the various scales are reproduced in the drawings exhibited, but are not shown in figs. 1 to 3, which indicate the way in which the rule is built up. Fig. 1 shows a side view, fig. 2 one end of the rule, and fig. 3 a section of the slide. The outer part consists of four strips of wood W, square in section, securely fixed parallel with each other by means of two brass pieces B, to which they are pinned. The inner space is occupied by the slide, which is formed of two pieces V pinned to an oblong piece A. The brass at one end of the rule bears two stars on one of its faces ; corresponding stars at the end of one face of the slide serve to indicate the correct way of inserting the slide in the rule.

For the purpose of the description of the scales the four faces of the instrument are in the drawings distinguished by the numbers 1, 2, 3, 4. When the slide is in its normal position there are on each face four scales, designated *a*, *b*, *c*, *d*, as shown in fig. 1.

The middle parts of the inner edges of the end brasses of the rule (*i.e.* the parts under which the sliding scales pass) are bevelled. A brass binding piece C can slide from one end of the rule to the other. It is an interesting fact that this piece can be used as a " cursor " or " runner." If this had been intended, the date of the actual introduction of the runner would be taken back 120

years.[1] The edges of this piece C are, however, not bevelled, as are the edges of the brass ends, and it is probable that it was intended to prevent the four wooden strips of the rule from bending outwards, as they are liable to do under end pressure, especially when the slide is withdrawn.

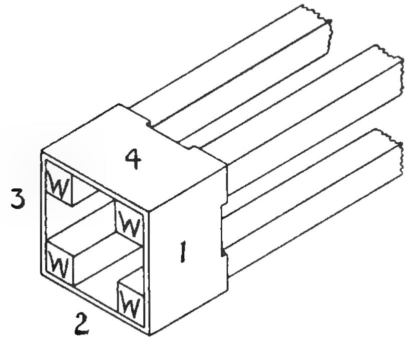


FIG. 2.

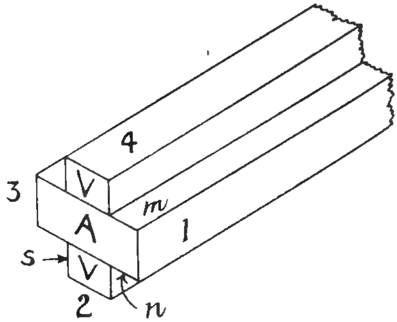


FIG. 3.

When the end of the slide bearing two stars is inserted in the side of the brass end with two stars, and passed along so that the scales a_1, b_1 coincide, the instrument is as shown in fig. 1, and the scales are as follows :—

(a_1), (b_1), (c_1). Gunter's *line of numbers*, doubled; each line being nearly a foot long. (The exact length is $11\frac{15}{16}$ inches. The original length would be 12 inches, according to the standard in use in 1654. Some of the difference will also be due to shrinkage during the 260 years which have elapsed since the rule was made.) The first number 1 is situated about half an inch outside the brass, so that it does not appear on scale a_1.

In the first half (1 to 10) of the scale each unit is divided into ten parts. In the second half, from number 1 to 3, each of these tenths is again divided into ten parts; from number 3 to 6 into five parts; and from number 6 to 10 into two parts.

(d_1) Gunter's "S.R." line or *sines of the rhumbs*. Numbered 1, 2, 3, 4, 5, 6, 7, 8. Each space divided into four parts.

(a_2) Gunter's *line of artificial sines*. Numbered 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 15, 20, 25, 30, 35, 40, 45, 50, 55, 60, 70, 80, 90. From 0 to 30 each degree is divided into six parts, from 30 to 50 into four parts, and from 50 to 70 into two parts; from 70 to 80 the degrees are not subdivided, and the space from 80 to 90 is divided into lengths of two degrees.

(b_2) Identical with a_2.

(c_2) Gunter's *line of artificial tangents*. Numbered

10, 15, 20, 25, 30, 35, 40, 45

89, 88, 87, 86, 85, 84, 83, 82, 81, 80, 75, 70, 65, 60, 55, 50,

Each degree is divided into six parts.

[1] According to Cajori, this useful addition to the slide rule was first made by Robertson, about 1775, though some contrivance of this kind had previously been suggested by Newton and Stone.

(d_2) Identical with c_2.

(a_3) Gunter's *line of meridians.* Numbered

01, 0z, 0Ɛ, 0ᔭ, 0ϛ, 09, 0L, 08, ϛ8

each degree being divided into four parts.

(b_3) *Line of equal parts,* two feet long. Numbered

0, 01, 0z, ... to 00z

each unit being divided into four parts.

(c_3) *Line of equal parts,* two feet long. Numbered

400, 390, 380, ... 20, 10, 0,

each unit being divided into two parts.

(d_3) Identical with d_2.

(a_4) ,, ,, a_1.

(b_4) ,, ,, c_3.

(c_4) ,, ,, b_4.

(d_4) ,, ,, a_2.

(m) Gunter's *tangent line of the staff,* 18 inches long. Numbered

zz, ϛz, 0Ɛ, ϛƐ, 0ᔭ, ϛᔭ, 0ϛ, ϛϛ, 09, ϛ9, 0L, ϛL, 08, ϛ8, 06

To 50, each degree is divided into six parts, from 50 to 70 into four parts, and from 70 to 90 into two parts. The distance between the two outer sights of the "cross" used with this scale would be 8·736 inches. The division 90 is half this distance from the end of the slide.

(n) Gunter's *tangent line on the staff,* two feet long. Numbered

0z, ϛz, 0Ɛ, ... 08, ϛ8, 06

The division 90 is at the extreme end of the slide. The distance between the middle and outer sights of the "cross" used with this scale would be 8·736 inches.

(s)[1] Numbered 1, z, Ɛ, ᔭ, ϛ, 9, L, 8, 6, 01, 11, z1; each unit being divided into sixteen parts from 1 to 6, and into four parts from 6 to 12. The whole distance from 1 to 12 is 21 inches, and the scale is divided in such a way that the distance from 1 to 2 is twice the distance from 2 to 4, which is twice the distance from 4 to 8.

It will be seen from the above that the logarithmic lines *number, tangent,* and *sine* are arranged in pairs, identical and contiguous, one line in each pair being on the fixed part, and the other on the slide.

This instrument was made three years before Seth Partridge wrote the description of his *Double Scale of Proportion,* and eight years before this description was published. As Partridge describes no feature which is not embodied in this example of the instrument, it would appear that less credit is due to him for invention in connection with the slide rule than has hitherto been given. Another point of interest is that the scales are figured so as to be read from left to right or *vice versa,* and not up and down, as in Oughtred's *Two Rulers for Calculation* (1633), or in Partridge's *Double Scale* (1662).

[1] The letter s in fig. 3 has inadvertently been applied to the wrong face of the lower piece V. In the instrument the scale s is on the opposite face of V, adjacent to the face n.

(2) Exhibits by Lewis Evans, Esq.

(The dimensions are given in inches)

1. An "*Universal Ring Dial*" of gilt brass, on one side of which is a circular slide rule. About 1700.

2. *Brass rule* with sights at the end for use in surveying, and with logarithmic scales on the under side. Dimensions 20 by $1\frac{1}{2}$ by $\frac{1}{8}$; number scale about $8\frac{3}{8}$-inch radius. B. Scott *fecit*. (Other work of Benjamin Scott dated 1733.)

3. *Boxwood rule*, 36 by $1\frac{3}{8}$ by $\frac{3}{8}$, radius 17·27. The slide made about 1820, the rule itself being older (about 1720).

4. *Boxwood rule* (German), $11\frac{1}{2}$ by $1\frac{3}{16}$ by $\frac{5}{16}$, having one slide, to draw out only. Radius $10\frac{1}{4}$. With its original leather case. Date 1737.

5. *Wooden rule*, 24 by $1\frac{3}{4}$ by $\frac{3}{16}$, covered with logarithmic and other scales. Radius $11\frac{1}{4}$, full. About 1790.

6. *Boxwood two-foot rule*, jointed, 12 by $1\frac{1}{2}$ by $\frac{1}{4}$. In one limb are two adjacent slides with two $5\frac{1}{2}$-inch radius scales in sequence. Wood & Lort, new improved sliding rule, Birmingham. About 1840.

7. *Gauging rule* with four slides, 12 by 1 by $\frac{3}{4}$. Maker Dollond, about 1850.

8. *Boxwood rule*, 13 by $2\frac{1}{8}$ by $\frac{3}{8}$, with two adjoining slides between two fixed scales, all with similar scales consisting of two 5-inch radius scales in sequence. Frederick A. Sheppard, Patentee. Maker, Stanley, Great Turnstile, about 1880.

9. *Ivory rule*, two-foot rule jointed, 12 by $1\frac{1}{2}$ by $\frac{3}{16}$. In one limb is a slide with two 5-inch radius scales in sequence. J. Routledge, Engineer, Bolton, about 1880.

Special Rules for Paper Makers

10. *Boxwood rule* with ivory slides, 12 by $1\frac{3}{4}$ by $\frac{3}{16}$, having two single slides and a pair bridged together. The scales are all $5\frac{1}{2}$-inch radius in sequence. Arranged by S. Waddington Barnsley. About 1890.

11. *Boxwood rule* with two adjacent slides, all scales 3·78, and in most cases three in sequence. Designed by S. Milne, Engineer. Patent protection No. 17794, 1891.

12. *Boxwood rule*, 38 by $1\frac{1}{4}$ by $\frac{1}{4}$, with one slide $16\frac{1}{2}$ radius, one and a half in series. Designed by L. Evans, 1891.

13. *Boxwood rule*, $20\frac{1}{8}$ by $1\frac{3}{8}$ by $\frac{3}{8}$, having one slide and a metal index. The upper scale is nearly $9\frac{7}{8}$, radius 25 cm., and the lower scale radius 50 cm. long. Tavernier-Gravêt, Rue Mayet 19, Paris. In a mahogany box. About 1900.

(3) ANDERSON'S PATENT SLIDE RULE. Exhibited by Brigadier-General F. J. ANDERSON. (Formerly manufactured by Messrs Casella & Co., London.)

In this rule the logarithmic scale from 1 to 10 is divided up into four sections on the upper part of the stock and on the slide, and into eight sections on the lower part of the stock. The slide carries two indices of transparent celluloid extending over the face of the rule to enable index settings and readings to be made on any part of the scale. The rule is operated like a standard slide rule, but with the following main exceptions:—(1) the lower scale is not repeated on the slide like the C scale, so that ordinary multiplication and division are confined to the *upper* scales and slide, and results are four times as accurate as those on the C, D scales of a standard rule of the same length; (2) as the same setting applies to four scales, the required scale is determined by means of a "line number" marked both on the stock and on the slide. In multiplication and division these "line numbers" follow the laws of logarithms, as they are virtually characteristics, but if the *right-hand* index be used, 1 must be added to the "line number" for a product; (3) the square root of a number on any part of the upper scale is read on the section of the lower scale bearing the same "line number," and similarly for squares. Forms a^3, a^2b, etc., are also calculable with the aid of the "line numbers."

FIG. 4.

(4) RAM PUMP CALCULATOR. Exhibitor, A. C. ADAMS, A.M.I.M.E.

1. This calculating slide rule has been designed with a view to facilitating the ready reckoning and checking of data in connection with ram pumps.

The top scale relates to discharge in cubic feet.

The second scale relates to discharge in gallons.

The third scale relates to time of pumping.

The fourth scale relates to revolutions, *i.e.* double strokes.

The fifth scale relates to length of stroke.

The left-hand side of the sixth scale relates to feet per second.

The right-hand side of the sixth scale relates to efficiency for single-acting, double-acting, and duplex types of pumps respectively.

The seventh scale relates to the diameter of the barrel in inches, *i.e.* from 3 inches to 30 inches.

The eighth scale is merely an extension to both ends of the seventh scale, *i.e.* 1 inch to 3 inches and 30 inches to 100 inches respectively. The latter section involves the use of two coefficients as applied to the duty. It is necessary to observe the settings on the right-hand side of the second scale, *i.e.* Oil, F., and S. These refer to Oil, Fresh Water, and Sea Water respectively.

Steam Engine Calculator

2. This calculating slide rule has been designed with a view of facilitating the ready reckoning and checking of data for steam engines.

The top scale refers to horse power, the second to piston speed, the third to length of stroke, the fourth to steam pressure per square inch, the fifth to cylinder diameter.

(5) Exhibit by W. E. Lilly, D.Sc.

Lilly's Improved Spiral Rule. (Exhibitor: W. E. Lilly, Trinity College,

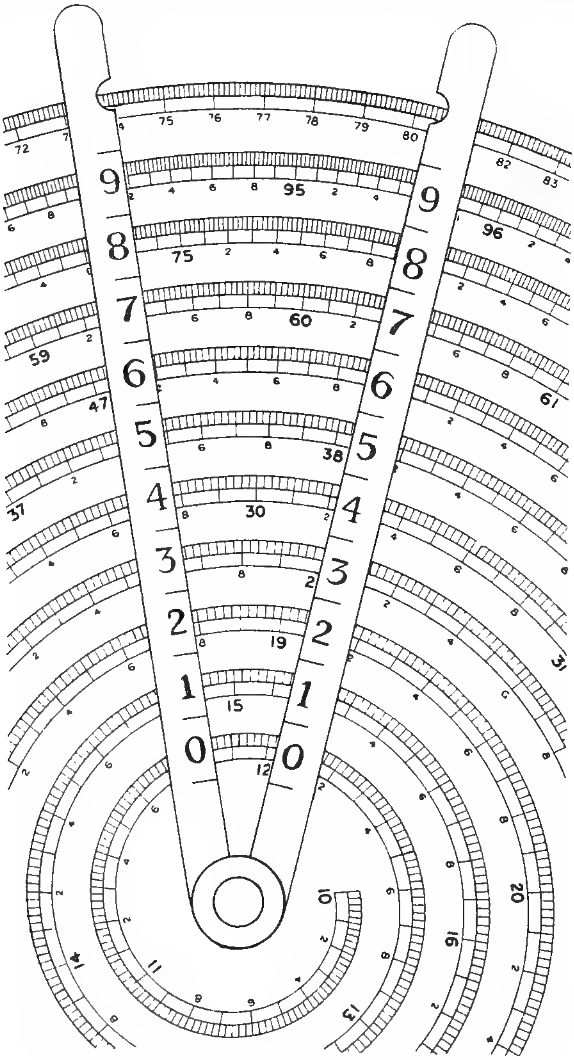


Fig. 5.—Actual size of Disc, 13·5 inches diameter.

Dublin.) This rule consists of a disc 13 inches in diameter with a spiral logarithmic scale of 10 convolutions, and a scale of 1000 equal parts on the outer edge for logarithms of numbers on the spiral. A pair of hands are mounted and held together by friction so as to be capable of any radial settings. This rule is equivalent to a straight rule about 30 feet long, and gives results correct to 4 figures.

(6) Exhibit by Dr Rudolph Taussig

The "Presto" Interest and Discount Calculator.—This calculator is designed to solve the formula $I = PRT \div 100$, and consists of three discs, the lowest (outermost) of which is fixed to the highest (innermost), while the middle

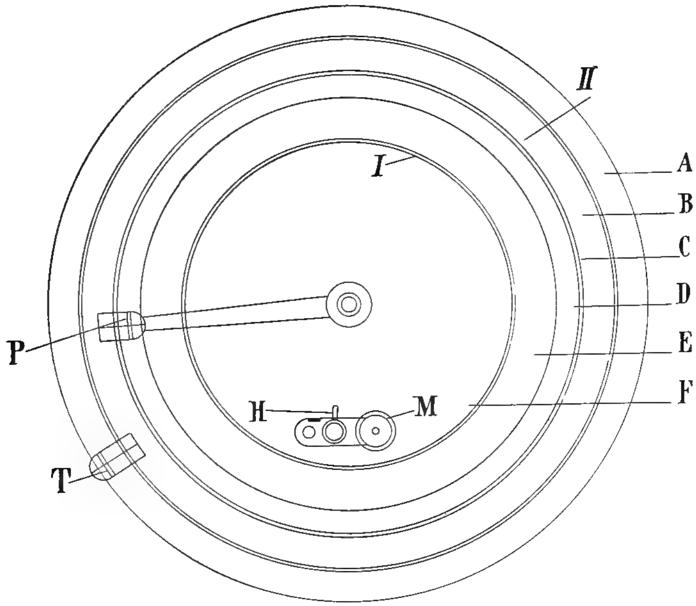


Fig. 6.

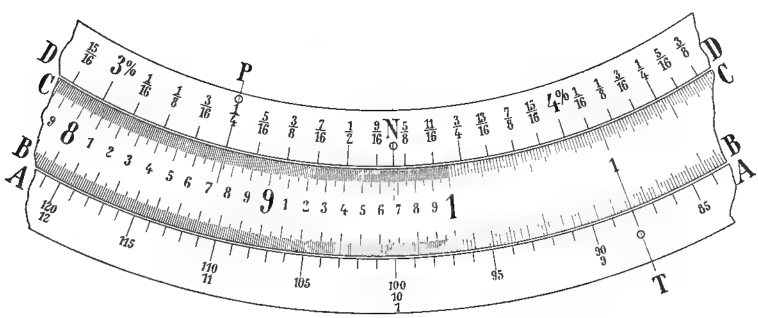


Fig. 7.

disc can be rotated about their common axis and set in any position. The outer ring carries the time scale A and is graduated to log 1/T, T in days from 20 to 200. The movable ring has a joint scale for P, the principal

(position B), and I, the interest (position C), being graduated to log x. The inner fixed ring on scale D gives rate per cent. (log R) from 1 to 10, subdivided into sixteenths of a unit.

In principle the calculator is an ordinary slide rule with scales modified to suit the special formula. The instrument commands all the accuracy called for in practice.

(7) A Patent Accessory to the Slide Rule. By R. F. Muirhead, D.Sc. (Extract from Provisional Specification.)

It consists in a method of combining with the slide-rule a mechanism by which roots and fractional powers of any number can be read off from the slide-rule scale. It depends on the principle that if the distance between the

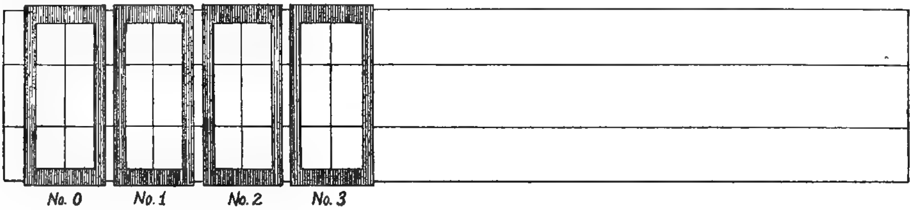


Fig. 8.—Plan of Top of Rule showing Cursors.

marks for 1 and for any number N on the slide-rule scale be divided into n equal parts, the end of the m^{th} part gives $N^{\frac{m}{n}}$, *i.e.* $\sqrt[n]{N^m}$ on the slide-rule scale.

The apparatus consists of a series of cursors which are movable relatively to one another along the slide rule, and are constrained, either by positive link mechanism of the "lazy-tongs" or other type, or by connecting springs, or otherwise, to remain equidistant from one another. The number of the cursors may be chosen at will, but it may be convenient to have seven of

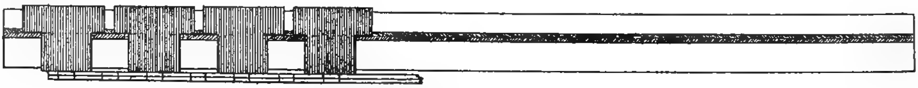


Fig. 9.—Side Elevation of Rule showing Connection between Cursors and Lazy-tongs Mechanism.

them, marked Nos. 0, 1, 2, 3, 4, 5, 6; and the scale of the rule may be from 1 to 100, as in the upper scale of the "Gravet" slide rule. Cursor No. 0 will be clamped to read 1 on the scale, and when close together, the cursors may occupy about half the length of the rule, so that cursor No. 6 reads 10.

If it is desired to read off, say, $N^{\frac{2}{5}}$ where N is a number lying between 10 and 100, then cursor No. 5 will be made to read N, and $N^{\frac{2}{5}}$ will be read off by cursor No. 2. This example indicates the method of using the accessory to read off fractional powers of any number between 10 and 100.

To deal with other numbers, the slider scale from 1 to 100 is divided into six equal intervals by cross-lines marked $\frac{1}{6}$, $\frac{2}{6}$, $\frac{3}{6}$, $\frac{4}{6}$, $\frac{5}{6}$, and also into five equal intervals by cross-lines marked $\frac{1}{5}$, $\frac{2}{5}$, $\frac{3}{5}$, $\frac{4}{5}$, and the manner of using it is indicated by the following example. To read off $23500^{\frac{3}{5}}$, which is $(23\cdot5\times10^3)^{\frac{3}{5}}=$

$(23{\cdot}5^{\frac{3}{5}}) \times 10^{\frac{9}{5}} = (23{\cdot}5)^{\frac{3}{5}} \times 10^{\frac{4}{5}} \times 10$, we pull out the slider so that the cross-line $\frac{4}{5}$ may be at 100 of the rule. Then make cursor No. 5 read 23·5 on the rule scale, and read off the digits of $23500^{\frac{3}{5}}$ on the slider scale, by means of the cursor No. 3.

With seven cursors we can thus read off the values of $N^{\frac{1}{6}}$, $N^{\frac{1}{3}}$, $N^{\frac{1}{2}}$, $N^{\frac{2}{3}}$, $N^{\frac{5}{6}}$, $N^{\frac{1}{5}}$, $N^{\frac{2}{5}}$, $N^{\frac{3}{5}}$ and also N^6, N^3, N^2, $N^{\frac{3}{2}}$, $N^{\frac{6}{5}}$, N^5, $N^{\frac{5}{2}}$, $N^{\frac{5}{3}}$, $N^{\frac{5}{4}}$, if these latter lie within the range, N being any number.

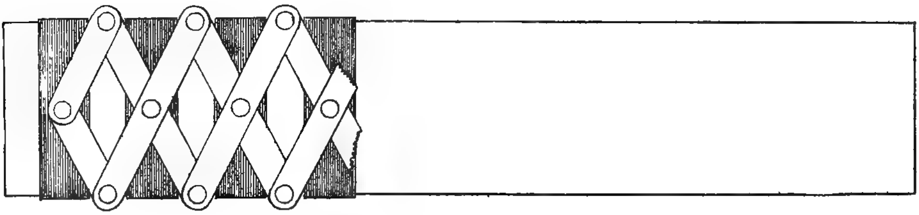


FIG. 10.—Plan of Rule reversed showing Lazy-tongs Mechanism for keeping Cursors equidistant.

It may be convenient to have cross-lines on the rule scale as well, to facilitate computations.

The accessory just described will apply to pocket slide rules or to larger ones for office use. A modification of the method would apply to disc calculators on the slide-rule principle.

(8) UNIVERSAL PROPORTION TABLE. By J. D. EVERETT, D.C.L., F.R.S.E. Lent by J. M. WARDEN, ESQ.

The date of publication is not stated, but Dr Everett was at the time assistant to the Professor of Mathematics in the University of Glasgow.

The table was designed to allow of multiplication, division, etc., being performed by inspection, with a result sufficiently accurate. It consists of two cards, A and B, one in the form of a grid, which correspond to the fixed and movable parts of a slide rule 160 inches long. These cards are divided accurately to scale, and when one card is properly laid upon the other, the portion common to both constitutes a complete table of proportional numbers for any ratio desired.

The table appears to be mainly useful for finding a fourth proportional.

For other operations—ordinary multiplication, division, or the finding of a reciprocal—all that is necessary is to take unity as one or other of the factors.

(9) SLIDE RULES designed by AUGUSTE ESNOUF, A.C.G.I.

The object for which these slide rules were designed is calculation dealing with construction in reinforced concrete. The principle employed is a development of that used for computations such as $Z = Kx^m y^n$. These rules determine the value of Z when given by the equation $Z = Kf(x^m y^n)$ for certain particular forms of f.

Two forms of slide-rule have been designed :—

1. *The Concretograph*

This is to deal with the complete design of reinforced concrete slabs and beams.

2. *The Struttograph*

The object of this instrument is to determine the load which a strut or column can sustain safely.

(10) Exhibits by E. M. Horsburgh, M.A.

1. Eighteenth-century boxwood rule, $9 \cdot 4 \times 1 \cdot 9$ inches. It is brass-bound at the ends, has two slides and twenty scales.

2. *Perry log-log rule,* 10 inch, with log-log scales E, F in addition to the standard scales. F measures log-log x from 1·1 to 10000, and E gives reciprocals of these numbers to enable a^{-n} to be read off. These scales are used in conjunction with the B scale. Makers: A. G. Thornton, Limited, Manchester.

3. *Proell's Pocket Calculator* consists of two cards, the lower of which carries the logarithmic scale in 20 sections, and the upper a similar scale on transparent celluloid, and running in the reverse direction. It is operated as an ordinary slide rule with the slide reversed. For continued multiplication and division, a needle (supplied with the instrument) is used as a substitute for a cursor, to fix the position of the intermediate results. A series of index points on the lower card enable square and cube roots to be extracted very easily. Makers: J. J. Griffin & Sons, Limited, London.

4. *Hudson's Shaft, Beam, and Girder Scale* gives at sight: the load a cast-iron, wrought-iron, or steel shaft will carry with any factor of safety; the diameter of a cast-iron, wrought-iron, or steel shaft to carry a given load; the load a beam or girder will carry at any span and factor of safety; the area required for a beam with a given span, load, and factor of safety, etc. Makers: W. F. Stanley & Company, Limited, London.

5. *R. H. Smith's Calculator,* similar to Fuller's rule in design and mode of operation. The scale line, 50 inches long, is wrapped round the central portion of a tube which is about $\frac{3}{4}$ inch in diameter and $9\frac{1}{2}$ inches long. A slotted holder, capable of sliding on the plain portions of this tube, is provided with four horns, these being formed at the ends of the two wide openings through which the scale is read. An outer ring carrying two horns completes the arrangement.

6. *Standard rules,* (*a*) 5·6 by 1·3 inches, made of cardboard. Maker: Gebrüder Wichmann, Berlin. (*b*) 10-inch. Maker: Thornton, Manchester.

(11) Exhibit from the Department of Electrical Engineering, University of Glasgow

Callender's Slide Rule for determining the Sizes of Cables.—This combination of slide rule and chart gives the size of cable required for transmitting electric power under given conditions of system of supply

(k), voltage (v), power (w), length of route (y), power factor (f), and percentage loss of voltage (p). The notation refers to the formula stated on p. 162.

(12) Exhibit from the Engineering Department, University of Edinburgh

Large Tavernier-Gravet Slide Rule, 6 feet 10 inches × 8 feet 5 inches.

(13) A Saccharometer and Slide Rule. Exhibited by John M. Maclean, B.Sc.

This instrument is a species of hydrometer used by brewers and officers of the Excise to determine the density of wort, the unfermented infusion of malt, which, when fermented, becomes beer.

The saccharometer and a thermometer are immersed in the wort, and the readings of both are taken. The reading of the saccharometer gives the correct density when the temperature of the wort is 60° F. If the temperature of the wort is not 60° F., the correct density may be obtained by means of the special slide rule supplied with the instrument.

For example, suppose the temperature to be 80° F., and that the saccharometer reading indicates the strength as 20. To find the correct strength by means of the slide rule, place the fleur-de-lis opposite 80° on the temperature scale, which is the portion of the rule graduated from 50° to 130°. Then opposite 20 on the slide the reading on the rule will be found to be 22·5, the correct density of the wort at 60° F.

The saccharometer readings give the excess weight per unit volume of the wort, taking the density of water at 60° as 1000. This explains the small rate of decrease of the length of the divisions of the slide.

The instrument was invented by Professor Thomson of Glasgow, and in 1816 an Act of Parliament enacted that it should be used by the Excise.

(14) An Improved Slide Rule. Exhibit by Professor E. Hanauer.

The feature of this form of the slide rule is the introduction of a scale marked on the slide, which is the same as the fundamental scales, but in the reverse direction. With this reciprocal scale it is possible to multiply or divide two numbers in two different ways with one setting of the instrument—thus providing a check, while the process of continued multiplication and division may be performed with fewer manipulations than are necessary on the ordinary type of slide rule.

The instrument is the design of Professor E. Hanauer of Budapest.

(15) W. F. Stanley & Company, Limited, Glasgow

1. *Thacher's Rule,* consisting of two logarithmic scales, one on the internal cylinder, and the other mounted continuously on the external bridges.

This rule is worked in the same manner as the ordinary straight slide rule, and gives results in 4 figures exactly. (Fig. 11.)

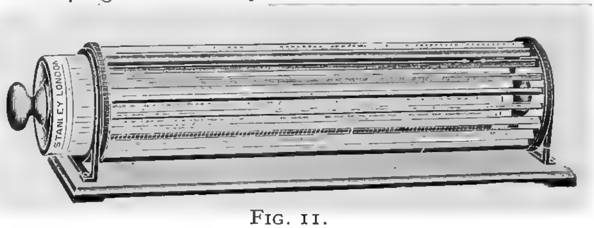


FIG. 11.

2. *Fuller's Rule* consists of a cylinder movable both round and parallel to its axis on a cylindrical stock to which a fixed index and a handle are attached. Another cylinder capable of telescopic and also rotational displacement lies within the stock and carries another index. A logarithmic scale 83 feet long is wound in a helix round the cylinder. Logarithms of

FIG. 12.

numbers on the scale are read on a scale of equal parts on the upper edge of the cylinder, in conjunction with the upper index. Tables of trigonometric functions are printed on the stock. Calculations are correct to 4 and sometimes to 5 figures. (Fig. 12.)

3. *Barnard's Calculating Rule*, similar to Fuller's, but the logarithmic scale is repeated twice and occupies in all only about one-third of the helix. The upper part of the helix carries a sine scale. Logarithms of numbers and sines are read as in Fuller's rule.

4. *Boucher's Pocket Calculator*, about the size of an ordinary watch, and equivalent to a 10-inch slide rule. It has scales on both faces. Those on the front give logarithmic numbers, sines and squares, or square roots. Those on the back give scale of equal parts, cubes and cube roots. (Fig. 13.)

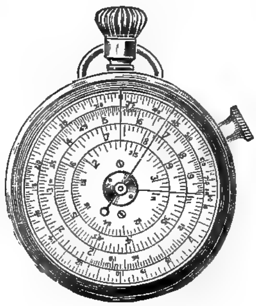


FIG. 13.

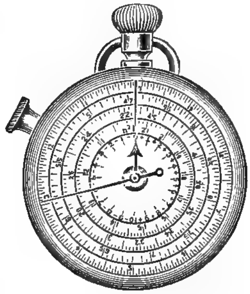


FIG. 14.

5. *Stanley-Boucher Calculator* is an improvement on the above by the addition of a third index hand on the back dial, which indicates the total movement of the front dial, so that continuous workings show a final result, either + or −, thus indicating the correct reading of the result. (Fig. 14.)

6. "*Rietz*"—10-inch standard rule, with scales E, F in addition, giving logarithms and cubes of numbers on D. The sides carry cm. and inch scales.

7. "*Precision*"—10-inch rule, designed to give the accuracy of a 20-inch rule. The logarithmic scale is in two sections: numbers on A, B run from 1 to $\sqrt{10}$, and on C, D from $\sqrt{10}$ to 10. The S and T scales are each in two sections also: in the upper sine section $\sin^{-1}\frac{1}{10}$ to $\sin^{-1}\frac{1}{\sqrt{10}}$ is read on B, in the lower section $\sin^{-1}\frac{1}{\sqrt{10}}$ to $\sin^{-1} 1$ is read on C. Similarly for tangents. On the bottom side of the rule are marked angles ($1° 49'$ to $5° 44'$) whose sines are read on D. An inch scale is given on the top side.

8. "*Universal*"—10-inch rule, designed for tacheometrical calculations. Scales A, C, D have numbers from 1 to 10, E from 1 to 100. F gives logarithms of numbers on A. B is a special scale in two parts: on the left log ($\sin x \cos x$) from $5° 50'$ to $45°$, continued in the middle of the slide from

5° 50′ down to 10′, on the right log ($\cos^2 x$) from 45° to 0°. S and T scales as in the standard rule.

9. "*Fix*"—10-inch rule, for mensuration of round bodies. Standard except in one respect. A is the standard scale in design, but displaced $\frac{\pi}{4}$ to the left relative to the stock. Opposite a reading d on D now stands $\frac{\pi}{4} d^2$ on A.

10. *Slide Rule for Chemists*, 10-inch, with scales C, D as in the standard rule. On A, B are a series of gauge points measuring logarithms of atomic and molecular weights to the same unit as on C, D. Another group of substances is given on the back of the slide.

11. *Hudson's Horse Power Computing Scale.*—A two-slide rule giving the I.H.P., the size of engine for a given power, the piston speed due to any stroke and number of revolutions per minute, the ratio the high- and low-

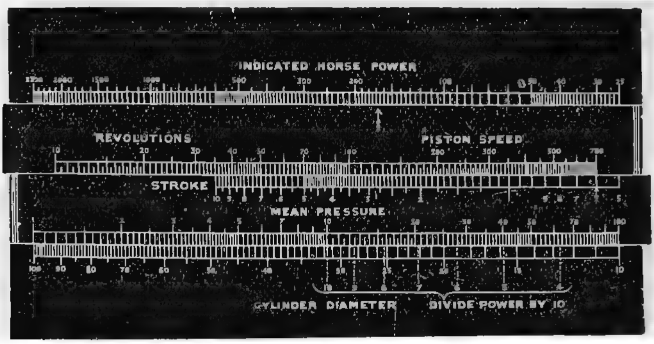


Fig. 15.

pressure cylinders of compound engines bear to each other, and the proportion the "mean" bears to the "initial" pressure. (Fig. 15.)

12. *The Essex Calculator for the Discharge of Fluids from Pipes, Channels, and Culverts*, designed to enable the engineer to ascertain rapidly and with fair accuracy the rates of velocity and discharge from sewers and water mains.

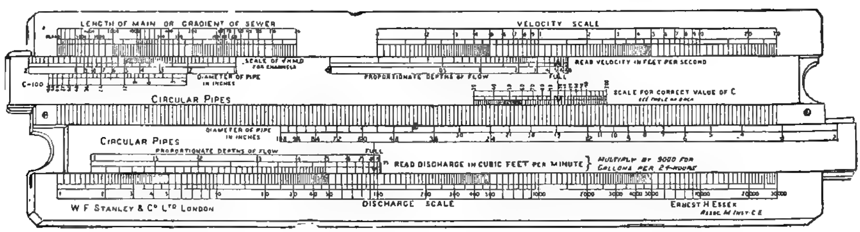


Fig. 16.

It can also be used to find the velocity of discharge in different forms of channel. The calculator is adjustable to the different formulæ in use, a scale for the value of c, the variable coefficient in Chezy's original formula $V = \sqrt{rs}$, being included on the upper slide and used in conjunction with a table of values of c on the back of the calculator. (Fig. 16.)

(16) John Davis & Son, Limited, Derby.

(All the following rules have celluloid facings and glass cursors.)

1. "*Simplex*"—standard rule (5-inch), containing scales A, B, C, D, with two bevelled edges.

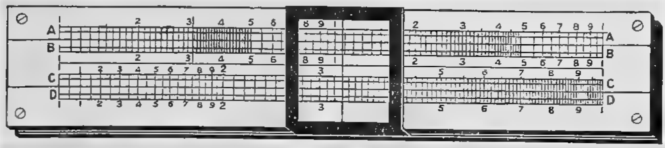


Fig. 17.

2. "*Simple*"—standard rule (10-inch), similar to "Simplex," but also divided on the edges in inches and mm. (Fig. 17.)

3. "*Hellen*"—standard rule (5-inch), containing scales A, B, C, D, S, T, and subdivided similarly to the 10-inch rule; also divided on the back in inches and mm., with magnifying cursor.

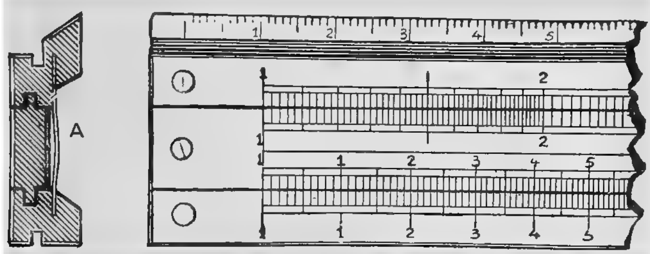


Fig. 18.

4. "*Hellener*"—standard rule (10-inch). The body is provided with a steel back as shown at A and is slotted under the slide longitudinally to overcome expansion or contraction. (Fig. 18.)

5. "*Special*"—standard rule (10-inch), with steel back as in "Hellener." There are also three adjusting screws to make the slide travel smoothly. For use in hot or damp climates.

6. "*Specialist*"—standard rule (20-inch), similar to "Special," but with five adjusting screws.

7. "*Onesee*"—10-inch rule, with scales A, B, C, D, E, F. E gives logarithms, and F gives cubes of numbers on D. (Fig. 19.)

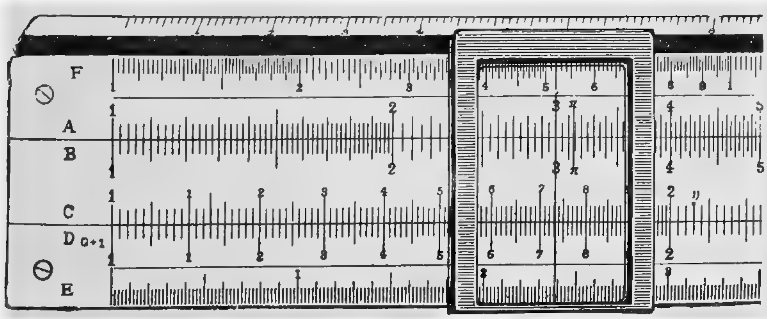


Fig. 19.

8. "*Oneseet*"—10-inch rule, similar to "Onesee," but with adjusting screws.

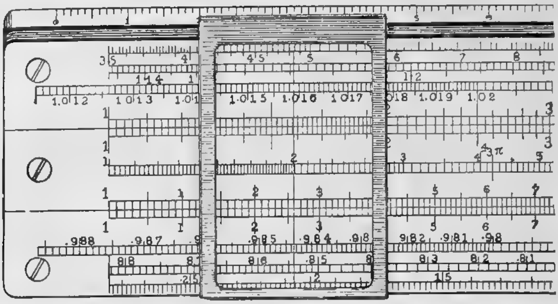


FIG. 20.

9. "*Yokota*"—10-inch rule, with seven scales on the face in addition to A, B, C, D. Three in the F position are sections of one log-log scale used in conjunction with scale C; three in the E position give reciprocals of numbers on these; and the seventh (on the slide) enables the 3rd, $\frac{2}{3}$rd, and $\frac{3}{2}$th powers of a number to be read off directly. Logarithms of numbers on D are read on the 10-inch scale on the side of the rule by means of a tongue on the cursor. (Fig. 20.)

10. "*Dunson*"—10-inch rule, with spare log-log slide, used in conjunction with scale C.

11. "*Stelfox*"—5-inch rule, with a 10-inch jointed slide, combining the accuracy of the larger with the portability of the smaller; the A, B, C, D

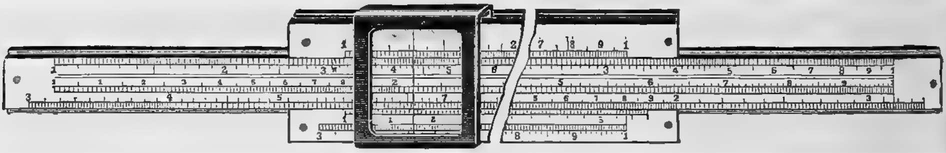


FIG. 21.

scales are as open as in the 10-inch size. The slide is jointed in the middle by means of long dowels, and can be separated instantly for carrying. The back of the rule and also of the slide are left blank for notes, which can be permanently or temporarily marked. (Fig. 21.)

12. "*Electrical*"—10-inch rule. With this new type of rule practical electrical calculations can be most simply and quickly carried out. One or two movements of the slide are usually sufficient, whereas, with the old type of rule, several settings were needed to obtain the required result. The scales on the edge of the slide, and those on the stock adjacent to them, are as in ordinary slide rules, so that the use of the instrument for the usual calculations is not interfered with in any way.

13. "*Jakins*"—11-inch rule. This is a quick and convenient instrument for performing calculations in surveying, and is a most ingenious device. It is claimed to give an average accuracy of within 1 in 10,000 in all calculations performed on the rule: at times the degree of accuracy greatly exceeds this. The rule is a distinct advance on all others, and although it is primarily intended for surveyors, its applicability is very wide.

(17) Group of "CASTELL" Slide Rules. Exhibited by A. W. Faber (London).

(All the following rules have celluloid scales and glass cursors.)

1. "CASTELL" standard rule (11-inch), with scales A, B, C, D, S, T, and in addition a scale between S and T giving logarithms of numbers on C.

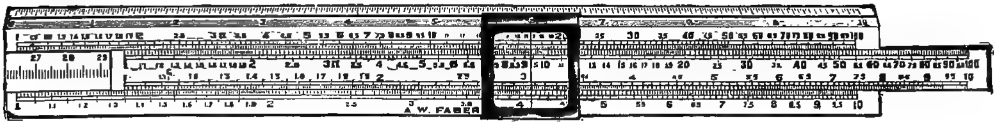


Fig. 22.

2. "CASTELL" standard rule (11-inch), including decimal, product, and quotient signs.

3. Ditto, but with registering cursor.

4. Ditto.

5. "CASTELL" standard rule (20-inch) with product and quotient signs.

6. Ditto, but with registering cursor.

7. "CASTELL" electrical and mechanical engineers' rule (11-inch), including a log-log scale in two sections E, F. F gives range 1·1 to 2·9, and E gives range 2·9 to 10,000. On the stock (beneath the slide) are two special scales, one for calculating efficiency of dynamos, effective horse-power, etc., and the other for loss of potential, current strength, etc. In other respects a standard rule.

8. Ditto, but $6\frac{1}{8}$ inches long.

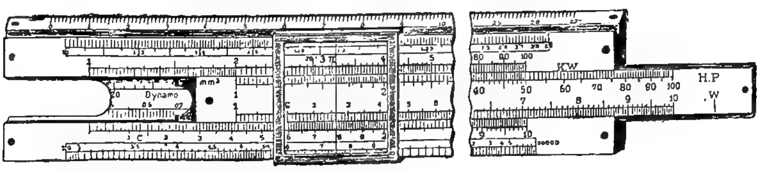


Fig. 23.

9. "CASTELL" standard rule (11-inch), with cube scale in addition.

10. Ditto, with registering cursor.

11. "CASTELL" standard rule (11-inch), including cm. and inch scales on the sides.

12. "CASTELL" standard rule (6-inch).

(18) Exhibit of Slide Rules. By A. G. Thornton, Limited (Manchester).

The case contains the following rules, which are all faced with celluloid and have aluminium glass cursors :—

1. "Perry" Patent Slide Rule. No. 6957. 10 inches long, $1\frac{1}{2}$ inches wide, with scale, inches, and fiftieths on bevel edge, cursor with pointer attachment, and divided line.
2. "Rietz" Pattern. No. 4908. $10\frac{5}{8}$ inches long.
3. Ordinary Patterns. Nos. 6039 and 4678.
4. "Technical" Slide Rule. No. 4977. 11 inches long.

PORTRAIT OF DR EDWARD SANG.

[*To face p.* 38.

SECTION G

OTHER MATHEMATICAL LABORATORY INSTRUMENTS

I. Integraphs. By CHARLES TWEEDIE, M.A.

§ 1. An Integraph may be briefly described as an apparatus for solving graphically a differential equation of the type

$$f\left(x, y, \frac{dy}{dx}\right)=0. \qquad . \quad . \quad . \quad . \quad . \qquad (1)$$

The machines invented naturally furnish solutions only for special forms of (1). Prominent among these is one for quadrature, when (1) is of the form $\frac{dy}{dx}=Q(x)$; so that

$$y=\int dx\, Q(x). \qquad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

Integraph of Abdank-Abakanowicz

It was for this purpose that Abdank-Abakanowicz (1878) invented the instrument that goes by his name, and which is the most familiar type of integraph. The theory of its construction is comparatively simple.

Consider a rectangular frame ABCD whose side BC can slide on the x-axis of a Cartesian system of axes. Take P on BC so that PC$=a$. Δ and I are two variable points moving on CD and AB in such a way that, as the rectangle is translated along the x-axis, Δ traces out a given curve ($Y=Q(x)$), while I is restricted to move so that the tangent to its path is constantly parallel to PΔ, and the co-ordinates of I are (x, y).

Now the gradient of PΔ at any instant is $Q(x)/a$; and that of the tangent at I is dy/dx.

Hence $$dy/dx=Q(x)/a,$$

and, for $a=1$, $$y=\int dx\, Q(x).$$

The two curves traced by Δ and I are called the differential curve and the integral curve ; and Δ and I are called the differentiator and integrator respectively. To I is attached a *rolling wheel*, whose plane is kept parallel to PΔ by a suitable mechanism. This rolling wheel is an essential part of all integraphs so far invented. From the use made of this, it will be spoken of in future as the integrating wheel.

For many years the integraph of Abdank-Abakanowicz was the only one in use, but in recent years, more especially through the researches of

Professor Pascal of Naples, numerous integraphs have been invented and constructed to solve differential equations of a more complicated character. We proceed to give some examples of these.

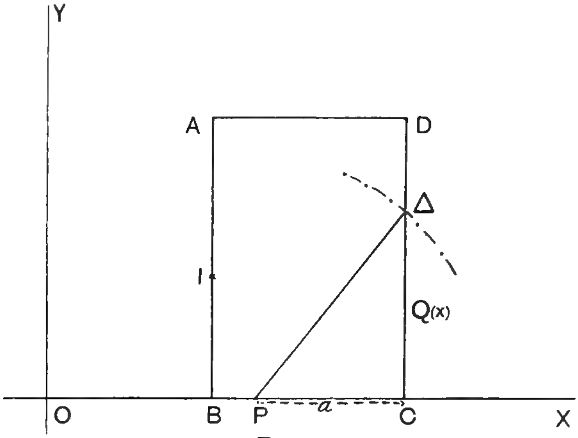


FIG. 1.

Integraph for the Linear Differential Equation $ay'+y=Q(x)$, in which a is a Constant

§ 2. Connect Δ and I on the rectangular frame by a rod ending in I and slotted for Δ. Let the integrating wheel attached to I be kept tangent to IΔ, so that the tangent to the integral curve traced by I is along IΔ.

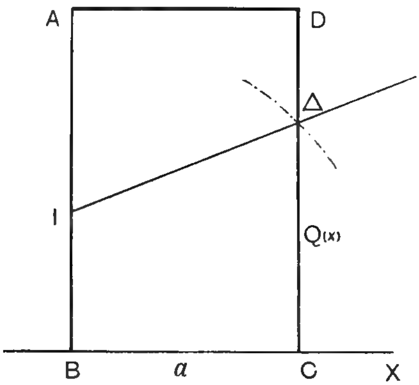


FIG. 2.

If, as before, I is the point (x, y), Δ the point of ordinate $Q(x)$, while $BC=a$, then the gradient of IΔ is

$$(Q(x)-y)/a.$$

Hence

$$y'=(Q(x)-y)/a,$$

or

$$ay'+y=Q(x). \qquad (3)$$

We note also that when IΔ is parallel to BC, $y'=0$, and the integral curve in general has a turning point. When IΔ is itself the tangent at Δ to the

differential curve, $y'=Q'(x)$, so that $y''=0$, and the integral curve has an inflexion.

Similarly, when the differential curve has a tangent parallel to the y-axis the integral curve has a cusp.

The general integral of (3) is

$$y=\frac{1}{a}e^{-\frac{x}{a}}\left(\int dx Q(x)e^{x/a}+C\right). \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The arbitrary constant C in (4) corresponds to the arbitrary position of I on AB when Δ is in the initial position on its graph.

Let y_1, y_2, y_3 . . . be the integrals corresponding to the values C_1, C_2, C_3 . . . of C. Then for the same value $x=\alpha$ of x, the ratio

$$\frac{y_1-y_2}{y_1-y_3}=\frac{C_1-C_2}{C_1-C_3}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and is therefore independent of x.

Hence if the integral curves cut two lines parallel to the y-axis in

$$A_1, A_2, A_3, \ldots$$

and

$$B_1, B_2, B_3, \ldots$$

the chords

$$A_1B_1, A_2B_2, A_3B_3 \ldots$$

are concurrent, for the two parallel lines are similarly divided.

In particular, when the two lines are taken infinitely near to each other, we have the theorem :—

The tangents to the integral curves, at points where they stream across a line $x=\alpha$, meet in a point.

This fact is directly obvious from the integraph; for no matter where I is taken on AB, the tangent must always pass through Δ. And it is an analytical consequence of the fact that the tangent to the integral curve through (α, η) has the equation

$$y-\eta=(x-\alpha)\frac{Q(\alpha)-\eta}{a}, \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and passes through the point $(\alpha+a, Q(\alpha))$, which is independent of η.

Cor.—When the integrating wheel makes a constant angle $\alpha=\tan^{-1}m$ with IΔ at I, the corresponding differential equation solved is

$$y'=\frac{Q(x)-y\pm am}{a\mp m(Q(x)-y)}. \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Integraph for a Canonical Form of Riccati's Equation

§ 3. B and E are fixed pivots on AB for two grooved bars BSΔ and ES.

The point S moves on these so that ES and IΔ are parallel, and the integrating wheel at I is directed along IS.

To find the differential equation solved, suppose, for a moment, the origin to be at B.

Let $BC=a$; $BE=b$; $\angle CB\Delta=\tan^{-1}m$.

Then Δ is the point (a, ma).
Let the co-ordinates of S be $(\lambda, m\lambda)$.

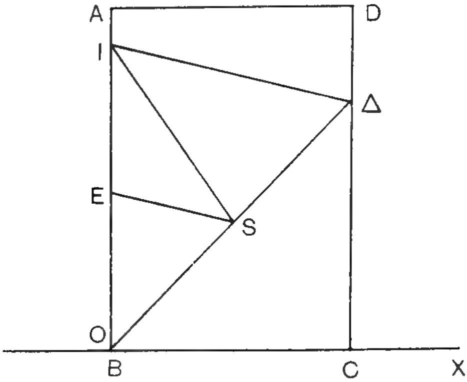


FIG. 3.

Since $BI/BE=B\Delta/BS=a/\lambda$

$$\therefore \quad BI=ab/\lambda,$$

and IS is the line

$$y=x(m-ab/\lambda^2)+ab/\lambda. \qquad (8)$$

Its gradient is $\therefore$

$$m-ab/\lambda^2=m-BI^2/ab.$$

Hence we obtain the differential equation

$$y'=m-y^2/ab$$
$$=\frac{Q(x)}{a}-\frac{y^2}{ab},$$

or

$$aby'+y^2=bQ(x). \qquad (9)$$

This is a canonical form of the equation of Riccati

$$y'=Ay^2+By+C, \qquad (10)$$

in which A, B, C are functions of x. At any point (α, η) on the line $x=\alpha$ the equation of the tangent, to the integral curve through it, is

$$y-\eta=(x-\alpha)(A\eta^2+B\eta+C). \qquad (11)$$

It contains η to the second power. Hence the tangents to the integrals as they stream across the line $x=\alpha$ envelop a conic section.

This is borne out by the integraph. For, at any instant, to any given position of Δ on CD, we can take any corresponding position for I on AB and the tangent is along IS. Now the rays ΔI and ES generate parallel and therefore projective pencils. Hence I and S generate projective ranges on BA and BΔ, so that IS envelops a conic, whose asymptotes are BA and BΔ.

§ 4. By taking a curved bar to connect Δ and I in § 2, Professor Pascal has shown how to obtain an integraph for $y'=F(Q(x)-y)$, where F is a known function of its argument; and more general results are obtained by replacing the guide AB on which I runs by a curved grooved bar connecting A and B.

He also gives an integraph suitable for the differential equation of the hodograph for the movement of a projectile in a resisting medium—a problem whose *analytical* solution is known only in a few cases.

All such integraphs in which Cartesian co-ordinates are used he classifies as Cartesian integraphs. When polar co-ordinates are used we obtain polar integraphs. (*Vide* Pascal, *I Miei Integrafi*, Naples, 1914.)

Polar Integraphs

§ 5. In the polar integraphs invented by Pascal, the fundamental rectangular frame is replaced by a circular sector AOB. The guides for Δ and I are OB and OA, and the sector has three supports: one at the centre

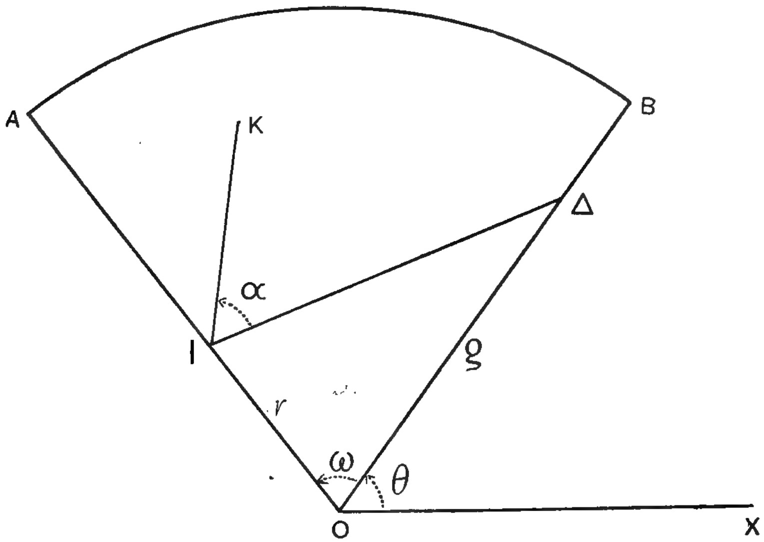


FIG. 4.

O, a heavy wheel at B, and the foot of a tracer at I. The instrument in use is rotated round O as a fixed point. The integrating wheel at I may make any constant angle α with IΔ.

Case (i)

Take $AOB=\pi/2$; $\alpha=\pi/2$.

Let $O\Delta=\rho$; $OI=r$; $XO\Delta=\theta$; and let $\rho=Q(\theta)$ be the equation to the path of Δ.

Then

$$\tan OIK=r\frac{d\left(\theta+\frac{\pi}{2}\right)}{dr}=r\frac{d\theta}{dr}.$$

Also

$$\tan OIK=-\cot OI\Delta=-r/\rho.$$

Hence

$$dr=-\rho d\theta,$$

and

$$r=-\int d\theta\, Q(\theta). \quad . \quad . \quad . \quad . \qquad (12)$$

The corresponding integraph is therefore suitable for quadratures.

Case (ii)

More generally, let AOB $=\omega$; ΔIK $=\alpha$; $\tan\alpha=m$.
Then $\tan \mathrm{OI}\Delta=\rho\sin\omega/(r-\rho\cos\omega)$.
Tan OIK $=\tan(\mathrm{OI}\Delta\pm\alpha)$.
Hence

$$r\frac{d\theta}{dr}=\frac{\rho\sin\omega\pm m(r-\rho\cos\omega)}{r-\rho\cos\omega\mp m\rho\sin\omega}. \qquad (13)$$

Thus, when $\alpha=0$, we obtain the equation

$$r'=\frac{dr}{d\theta}=\frac{r^2}{Q(\theta)\sin\omega}-r\cot\omega, \qquad (14)$$

which is an equation of Bernouilli (a linear equation in $1/r$).

Also if $\alpha=\omega$

$$r'=r\cot\omega-\frac{Q(\theta)}{\sin\omega}, \qquad (15)$$

a canonical form of the linear equation.

In his treatise Pascal uses (12) to obtain an abacus for each of the following functions :—

$$\int d\theta/\sqrt{(1-k^2\sin^2\theta)}\ ; \ \int d\theta\sqrt{(1-k^2\sin^2\theta)}\ ; \ \int d\theta/\cos^n\theta.$$

For further information consult Pascal (*l.c.*) and Galle's *Mathematische Instrumente* (Teubner, 1913) ; also *Les Intégraphes* of Abdank-Abakanowicz.

Integraph lent by the ROYAL TECHNICAL COLLEGE, GLASGOW, *per* Professor JOHN MILLER, D.Sc.

This instrument was manufactured by Coradi of Zürich and embodies the fundamental principle of Abdank-Abakanowicz. A vertical tracing wheel, whose projection on the drawing plane has always a gradient proportional to the ordinate of a given curve, traces out the integral of this curve. The whole instrument rolls without slipping on four roughened rollers in the direction of the axis of abscissæ. The tracing wheel is rigidly fixed in a frame (A), which runs on two wheels which move in a groove on the upper of two parallel bars of the instrument in the direction of the axis of ordinates. This bar is divided into millimetres, and with a vernier in the frame A gives readings for the area traced out. Another vernier in the frame A moves along a lower parallel bar divided into tenths of an inch and gives the same reading in inches. To the ends of an axis, parallel to the axis of the tracing wheel, in the frame A are attached two bars moving freely horizontally. These are attached to another bar parallel to the axis in the frame A, and the four form a freely deformable parallelogram. This fourth bar, by means of two wheels, runs in a groove on a guider which rotates horizontally on a pivot fixed in the upper bar on which the frame A moves. Thus, the guider is always parallel to the tracing wheel or at right angles to its axis. On the under side of this guider is a second groove. Into this groove fits an edge in an upright fixed to another frame (B), which also runs by wheels in grooves

on bars parallel to the axis of ordinates. This upright is movable along a scale parallel to the axis of abscissæ. This scale is graduated from 10 to 20 centimetres, or from 4 to 8 inches, and the alteration in the upright alters the scale of reading in the instrument, that is, the factor of proportionality in the integration. In a bar in the frame B is a pencil or point which traces out the original curve. This bar is movable parallel to the axis of ordinates and can also be reversed so as to bring the tracing point to the right or left. To the frame A are attached two pencils, either of which may trace out an integral curve. One is at the back near the tracing roller; the other is at the front, so that the abscissæ of corresponding points on the original curve and the integral curve are the same.

The following table gives the constants of the instrument :—

Base.	Values of a Centimetre (or Inch) of the Ordinate of a Curve which the Integraph draws when tracing out the following Curves.		
	First Curve (area).	First Integral Curve (first moment).	Second Integral Curve (second moment).
100 mm.	10 cm.^{2}	100 cm.^{3}	1000 cm.^{4}
160 mm.	16 cm.^{2}	256 cm.^{3}	4096 cm.^{4}
200 mm.	20 cm.^{2}	400 cm.^{3}	8000 cm.^{4}
4″	4″^{2}	16″^{3}	64″^{4}
5″	5″^{2}	25″^{3}	125″^{4}
8″	8″^{2}	64″^{3}	512″^{4}

II. Integrometers. By G. A. Carse, D.Sc., and J. Urquhart, M.A.

We propose here to deal briefly with the instruments known as Integrometers, following the French usage of the term Intégromètres: these instruments may also be called Moment Planimeters (Planimètres à moments), their object being to calculate, usually by a single operation, the three integrals $\int y dx$, $\int y^2 dx$, $\int y^3 dx$, and in some cases $\int y^4 dx$, taken over a given area.

The importance of these instruments from a practical point of view is that they enable centres of gravity and moments of inertia to be determined mechanically.

It is interesting to notice that Oppikoffer's planimeter[1] can be used to determine $\int y^2 dx$ by means of two operations. If the curve that M (see fig. 1, Planimeters) traces on the cone be considered, we see that its area is $\lambda \int y^2 dx$ where λ is a constant, for if M′ be a consecutive position of M, the area of the elementary triangle VMM′ is $\lambda y^2 dx$, and hence if this curve be traced on a sheet of paper wound round the cone, by unfolding the paper and finding by means of the planimeter the area between this new curve and the initial and final positions of VM, we can determine $\int y^2 dx$.

[1] Art. "Planimeters," this Handbook.

It should also be noticed that, given a curve $y=f(x)$, we can, by squaring the ordinates, trace the curve $y=f^2(x)$, and hence, finding the area of this new curve by the planimeter, we can calculate $\int y^2dx$, where $y=f(x)$, and clearly this method can be extended to finding $\int y^3dx$, $\int y^4dx$ . . .

If in fig. 6, Planimeters, the line C′ be taken to be the axis of x, then $y=l\sin\alpha'$, $y^2=l^2\sin^2\alpha'$, . . . $y^n=l^n\sin^n\alpha'$. Thus, instead of considering the integrals $\int ydx$, $\int y^2dx$, . . . $\int y^ndx$, we may consider the integrals $\int\sin\alpha'dx$, $\int\sin^2\alpha'dx$, . . . $\int\sin^n\alpha'dx$. We know that $\sin^n\alpha'$ can be expanded in terms of cosines of multiples of α' if n be even, and in terms of sines of multiples of α' if n be odd. Thus the original integrals can be shown to depend finally on integrals of one or other of the types $\int\sin m\alpha'dx$, $\int\cos m\alpha'dx$, where $m=n$, $n-2$ . . .

Now, if an integrating wheel with its axis making an angle $m\alpha'$ with the x-axis be attached to the arm of constant length, it will enable us to read off the value of $\int dx\sin m\alpha'$. Likewise, if we take a wheel making an angle $\left(\frac{\pi}{2}-m\alpha'\right)$ with the x-axis, we get the value of $\int dx\cos m\alpha'$.

Thus, since $\sin^2\alpha'=\frac{1}{2}-\frac{1}{2}\cos 2\alpha'$, $\sin^3\alpha'=\frac{3}{4}\sin\alpha'-\frac{1}{4}\sin 3\alpha'$, we have $\int dx\sin^2\alpha'=\frac{1}{2}\int dx-\frac{1}{2}\int dx\cos 2\alpha'=-\frac{1}{2}\int dx\cos 2\alpha'$ for $\int dx=0$, since the arm AB will return to its original position when A makes a complete circuit of the curve and $\int dx\sin^3\alpha'=\frac{3}{4}\int dx\sin\alpha'-\frac{1}{4}\int dx\sin 3\alpha'$. We thus see that in an instrument giving $\int ydx$, $\int y^2dx$, $\int y^3dx$ simultaneously we must have integrating wheels making angles α', $\frac{\pi}{2}-2\alpha'$, $3\alpha'$, with the x-axis. Amsler[1] devised an integrometer based on this principle. Amsler has also constructed an instrument giving in addition $\int y^4dx$, in which, as is clear from the general theory, the addition of a fourth integrating wheel making an angle $\frac{\pi}{2}-4\alpha'$ with the x-axis is necessary.

In the Amsler instruments the integrating wheel giving the area rolls on the paper, the remaining wheels rolling on discs as in disc planimeters (*q.v.*).

Improved instruments have been devised by Hele-Shaw and constructed by Coradi. In these, the integrating wheels roll on spheres, and thus any error due to inequalities of the paper is eliminated, and further, the wheels have a motion of rolling only.

Another integrometer is that of Desprez,[2] in which there is only one integrating wheel, which performs successively the various integrations.

[1] *Dyck's Catalogue*, p. 202, 1892.
[2] Morin, *Les Appareils d'Intégration.*

(1) **The Hele-Shaw Integrator.** Exhibited by G. Coradi, Zürich.

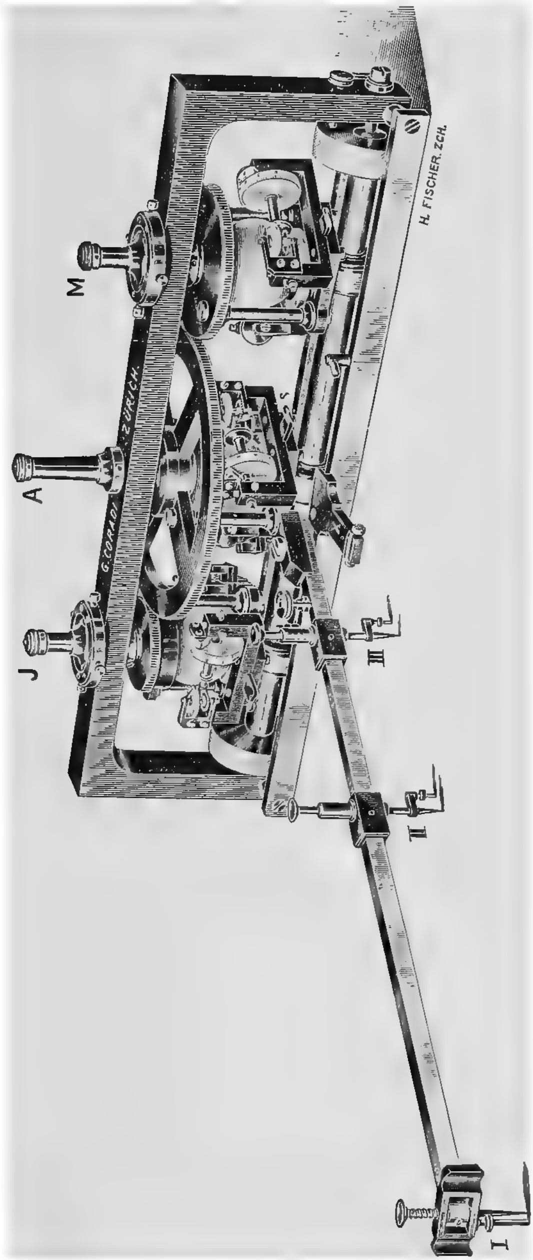


Fig. 1.

This integrator employs three glass spheres. It is used for determining the area, moment of stability, and moment of inertia of plane figures.

(2) **A Radial Integrator.** By E. M. Horsburgh.

III. Planimeters. By G. A. Carse, D.Sc., and J. Urquhart, M.A.

Numerous instances occur in the mechanical, physical, and biological sciences in which it is required to determine the area of a closed curve, got by a series of observations, which may be either continuous, as in some self-recording apparatus, or taken at successive intervals.

The necessity of repeatedly performing such a calculation gave rise to attempts being made to devise instruments called planimeters which would give the desired result rapidly, and to a considerable degree of accuracy.

Various types of planimeters exist, including some devised for special purposes, but the majority of instruments which are of practical value can be classified under two main types.

We propose to deal with instruments which fall under one or other of these types, reference, however, being made to special forms which are of general interest and importance.

The two types are :

I. Rotation planimeters, so called because the essential part of the apparatus consists in general of a wheel—the integrating wheel—rolling on a disc or cone which is itself capable of rotation.

II. Planimeters with an arm of constant length—of the well-known Amsler type.

Type I.—Rotation Planimeters

It is probable that J. M. Hermann[1] designed a planimeter about 1814, which two years later was improved by Lämmle, but as no description of the instrument was published at the time, it was overlooked, and does not appear to have had any influence on the evolution of the planimeter. Following Hermann's model, Gonella, in 1824, devised a planimeter, descriptions of which appeared in 1825[2] and 1841,[3] and these were the first publications relating to planimeters. In this instrument the integrating wheel rolled on a cone, which was replaced later by a horizontal disc. Owing to the difficulty of getting instruments accurately made at that period, Gonella was unable to get his design executed satisfactorily.

Ernst in Paris constructed instruments of practical use based on the design of Oppikoffer,[4] who about 1827 adopted the principle of a wheel rolling on a cone.

The essential parts of Oppikoffer's instrument consist of a cone which is capable only of rotating about its axis, and placed in such a position that a generator VM, say, is always horizontal ; a wheel R in contact with the cone with its plane perpendicular to, and its point of contact on, VM ; and a wheel R′ with its plane perpendicular to the axis of the cone and rigidly attached to it. The wheel R is capable of rotation about its axis and of sliding along the horizontal generator. The wheel R′ is made to rotate by

[1] *Dingler's Journal*, vol. cxxxvii.
[2] Gonella, *Teoria e descrizione*, etc., Florence, 1825.
[3] Gonella, *Opuscoli matematici*, Florence, 1841.
[4] Morin, *Les Appareils d'Intégration*, 1913.

resting on a horizontal rail in the plane of the wheel, the rail being capable of moving along its length.

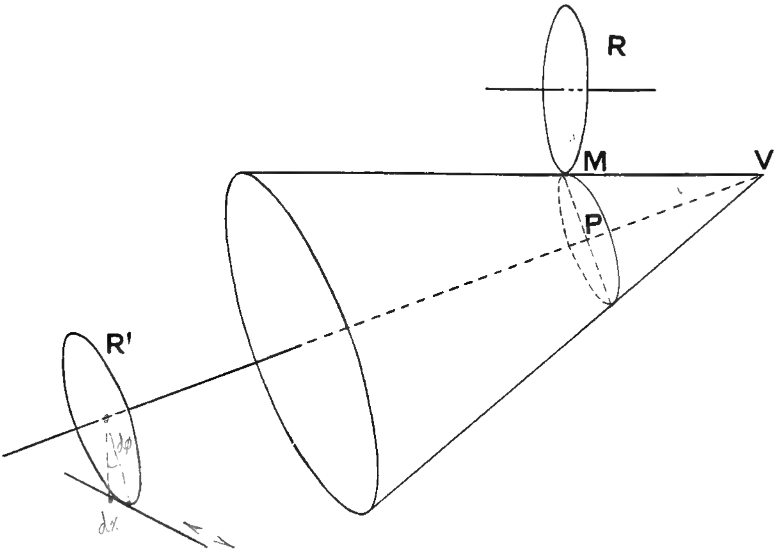


FIG. 1.

Suppose a line parallel to the rail is taken as x-axis, and a line perpendicular to the rail as y-axis. A pointer is made to trace the curve and is capable of motion in such a way that its distance from the x-axis, *i.e.* the y coordinate, is always equal to VM. This is attained by means of a mechanism such that a motion of the pointer in the direction of the y-axis only moves the wheel R along VM, while a motion of the pointer in the direction of the x-axis merely moves the rail. If the pointer traces a given curve, the length VM will be equal to the y coordinate of the curve.

If the rail receives a displacement dx, the wheel R′, and consequently the cone, rotates through an angle

$$d\phi = \frac{dx}{r'}$$

where r' is the radius of R′.

The displacement of M will therefore be

$$\text{PM}d\phi = y \sin\alpha\, d\phi, \text{ if } \alpha \text{ be the semi-vertical angle of the cone}$$

$$= y \sin\alpha \frac{dx}{r'}$$

$$= \frac{\sin\alpha}{r'} y dx.$$

But the displacement of M is $rd\omega$, where r is the radius of R and ω the angle turned through by R.

$$\therefore\ rd\omega = \frac{\sin\alpha}{r'} y dx$$

$$i.e.,\ \omega = \frac{\sin\alpha}{rr'} \int y dx.$$

We thus see that the angle turned through by R is proportional to the area of the curve traced by the pointer.

Early attempts by Wetli[1] in 1849 to improve planimeters resulted in the substitution of a circular disc for the cone, as we have already mentioned had been done by Gonella, and it is clear that the above theory obviously applies, for the cone may be degenerated into a circular disc by making $\alpha = \frac{\pi}{2}$. Improved Wetli instruments giving results to a considerable degree of accuracy are due to Starke of Vienna, and Hansen of Gotha.

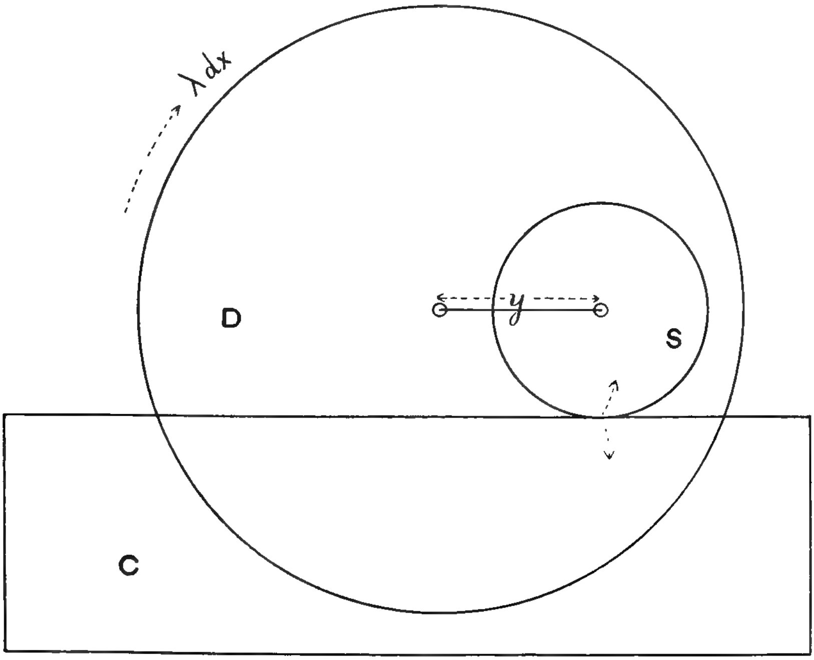


FIG. 2.

In the 1851 Exhibition in London, various types of planimeters were exhibited, among which was one by Sang.[2] The combination of rolling and slipping of the integrating wheel in all the above forms of planimeters is not entirely satisfactory, and Maxwell,[3] being struck with this imperfection in Sang's planimeter, devised a mechanism in which the sliding action was dispensed with. This he achieved by having two equal spheres rolling each on the other. At a later period J. Thomson[4] had his attention drawn to this matter, and he endeavoured to devise a method depending on pure rolling contact, which would render the mechanism simpler than that of Maxwell. He succeeded in devising a new kinematic principle on which he based a planimeter.

His mechanism consists of a disc D, sphere S, and circular cylinder C arranged

[1] Wetli, *C. R. de l'Acad. de Sci. de Vienne*, 1850.

[2] *Trans. Roy. Scot. Soc. Arts*, vol. iv., 1852. [3] *Ibid.*, vol. iv., 1855.

[4] Thomson and Tait's *Natural Philosophy*, vol. i., App. B[1] III., 1896; *Proc. Roy. Soc.*, xxiv. 262, 1876.

as follows. The disc is capable of rotation about an axis perpendicular to its plane and passing through its centre. The cylinder, which is not in contact with the disc, can rotate about its axis, which is parallel to the plane of the disc. The sphere is always in contact with the disc and the cylinder, and can roll along, keeping in contact with both and not making either rotate. The path of the point of contact with the disc is a diameter of the disc, while the path of the point of contact with the cylinder is a generator, and this diameter is parallel to the generator. If y be the distance between the point of contact of the sphere with the disc and the centre of the disc, and the disc receive a rotation λdx, where λ is a constant, in the direction of the arrow, the point of contact with the disc moves through a distance $\lambda y dx$, and the point of contact of the sphere with the cylinder also moves through the same distance. Hence, if r be the radius of the cylinder, the angle $d\omega$ turned through by the cylinder in the direction of the arrow is $\frac{\lambda y dx}{r}$. Thus the total angle turned through by the cylinder measures

$$\int y dx.$$

A mechanism can be devised which will enable the necessary rotation λdx to be given to the disc and make the sphere move so that its point of contact with the disc is at a distance y from the centre of the disc, x and y being the coordinates of a point *on* the curve whose area is required. With such a mechanism it will be possible to calculate the area by measuring the rotation of the cylinder.

It was pointed out to Professor J. Thomson by his brother Lord Kelvin, that the addition of a further piece of mechanism renders the machine capable of giving a continuous record of the growth of the integral. The mechanism required to be introduced for this purpose is such that it describes continuously a curve whose abscissa and ordinate at any point shall represent x and $\int_0^x y dx$ respectively. Kelvin's device consists of a second cylinder coaxal with and rigidly attached to the axis of the disc, and a rod parallel to the axis of the second cylinder, bearing on the first cylinder, and provided with a point which traces on a roll of paper on the second cylinder the curve $Y = K\int y dx$ where K is a constant. It is clear, therefore, that this arrangement can be used as an integraph. This planimeter forms the basis of Kelvin's Harmonic Analyser [1] and Tide Calculating Machine.

For other planimeters of the rotation type, Stadler,[2] Amsler,[3] and the Paris firm of Richard Frères [3] are responsible. The last of these is interesting in that it enables the area of a curve such as is given by a trace on a cylinder in a recording apparatus to be measured.

[1] *Proc. Roy. Soc.*, xxvii. 371, 1878; Thomson and Tait's *Natural Philosophy*, vol. i., App. B[1] VII., 1896.

[2] *Dyck's Catalogue*, 1892.

[3] Morin, *Les Appareils d'Intégration*, p. 61, 1913.

TYPE II.—PLANIMETERS WITH AN ARM OF CONSTANT LENGTH

The construction of this type is based on the following theory :—

Consider two areas S, S′, bounded by the closed contours C, C′. Take a point A on C and a point B on C′, and suppose that AB is of a constant length l. If the point A makes a complete circuit of the contour C, while B is constrained to move on C′, the area swept out by AB is equal to S−S′.

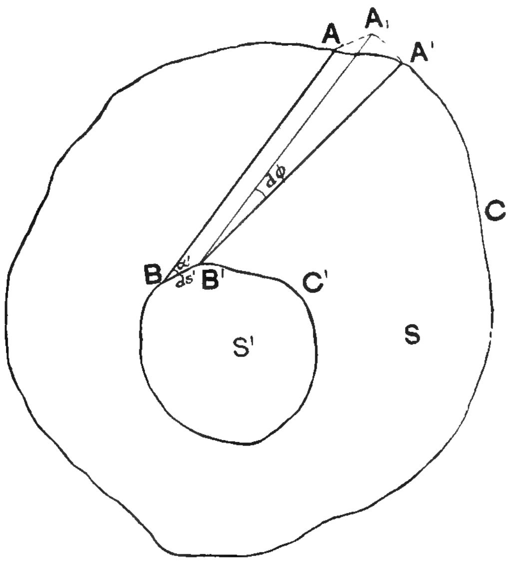


FIG. 3.

If in this motion AB be any position of the arm, and A′B′ a consecutive position, the elementary area ABB′A′ swept out by AB$=lds'\sin\alpha'+\frac{1}{2}l^2d\phi$ where ds' is the arc BB′ of C′, α' is the angle between AB and the tangent at B to C′, and $d\phi$ is the angle turned through by AB. This is seen at once by moving AB parallel to itself to the position $B'A_1$, and then rotating it about B′ through the angle $d\phi$, for the area is equal to the sum of the areas of the parallelogram $ABB'A_1$ and the triangle $A_1B'A'$. Hence, adding these areas, we get, when C′ is entirely exterior[1] to C, $S-S'=l\int ds'\sin\alpha'$, for $\int d\phi=0$, since the arm AB will return to its original position, and if C′ be entirely interior to C we get $S-S'=l\int ds'\sin\alpha'+\pi l^2$, for in this case the arm AB will have made a complete revolution, *i.e.* $\int d\phi=2\pi$.

The following device is used for measuring $\int ds'\sin\alpha'$. A wheel of radius r—the integrating wheel—is attached to AB produced, with its axis parallel to AB. As AB moves parallel to itself into the position $B'A_1$, any point in

[1] In the case of the Amsler Polar Planimeter C′ reduces to an arc of a circle.

the circumference of the wheel moves through a distance ds' sin α', and during the rotation from the position $B'A_1$ to the position $B'A'$ it travels through a distance $ad\phi$ in the opposite direction, where a is the distance of B' from the plane of the wheel. Hence if dn be the displacement of a point in the circumference of the wheel

$$dn = ds' \sin \alpha' - ad\phi$$

$$\therefore\ n = \int ds' \sin \alpha' - a \int d\phi$$

$$i.e.\ \int ds' \sin \alpha' = n + a \int d\phi$$

$= n$ if S′ be exterior to S

and

$= n + 2\pi a$ if S′ be interior to S.

This type has been subdivided into the following sub-classes, according to the nature of the guiding curve (C′) :—

(*a*) Polar, in which the guiding curve is a circle.
(*b*) Linear, in which the guiding curve is a straight line.
(*c*) Planimeters in which the guiding curve is not any curve in particular.

(a) *Polar Planimeters*

If another arm OB, called the polar arm, have the end O fixed at O, which is called the pole, and the other end jointed at B, the point B is constrained to move on a circle whose centre is O, when the end A of the arm of constant length traces the curve whose area is required. The best-known instrument of this type is Amsler's Polar Planimeter.

$S - S' = l \int ds' \sin \alpha' + \pi l^2$

$S - S' = l\,(n + 2\pi a) + \pi l^2$

$S - S' = ln + 2\pi l a + \pi l^2$

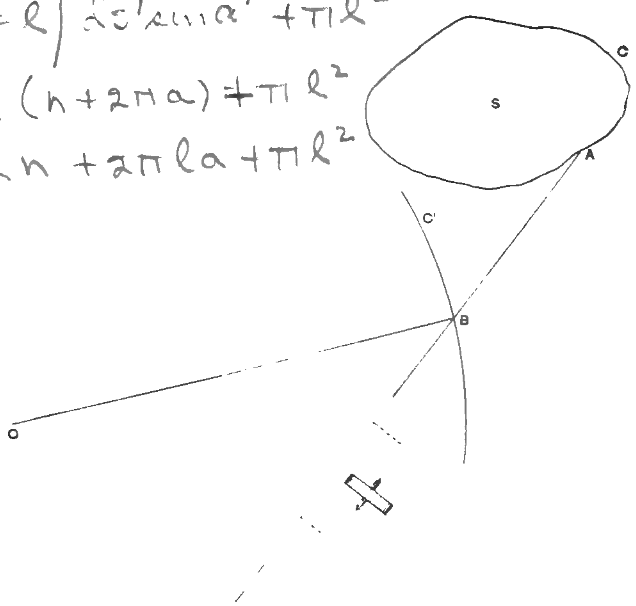


FIG 4.

If OB be equal to b, the general formula gives $S=nl$, or $S=nl+2\pi al+\pi l^2+\pi b^2$, according as the circle is entirely outside or entirely inside the area S.

If very accurate results are required, account must be taken of several sources of error. One of these errors is that due to the axis of the integrating wheel not being parallel to the arm AB. Various instruments called Com-

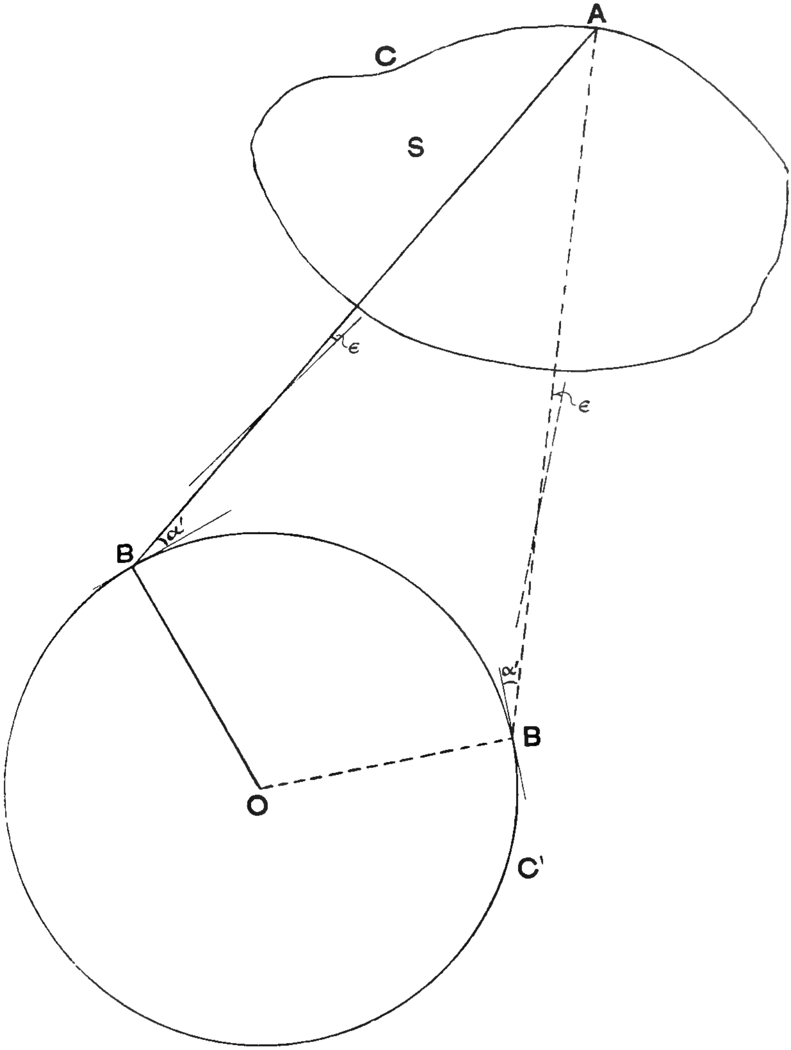


FIG. 5.

pensation Planimeters have been constructed, in which attempts have been made to eliminate this error. The method for eliminating the error was given by Lang.[1] In fig. 5, if the axis of the integrating wheel make a small angle ϵ with AB, the part of the reading recorded by the integrating wheel corresponding to $ds' \sin \alpha'$ is now $ds' \sin(\alpha'-\epsilon)$, when AB receives a small displacement.

[1] *Zeitschrift für Vermessungswesen*, 1894.

If now we consider the symmetrical position with respect to OA as indicated by dotted lines in the figure, the pole being kept fixed, the part of the reading of the wheel due to a small displacement of AB is $ds' \sin(\pi - \overline{\alpha' + \epsilon})$ instead of $ds' \sin(\pi - \alpha')$. Hence in tracing the curve, starting from the former position, we get $\int ds' \sin(\alpha' - \epsilon)$, while tracing the curve in the same sense, starting from the latter position, we get $\int ds' \sin(\pi - \overline{\alpha' + \epsilon})$, *i.e.* $\int ds' \sin(\alpha' + \epsilon)$. The mean of these two readings is $\frac{1}{2}\int ds' \left\{ \sin(\alpha' - \epsilon) + \sin(\alpha' + \epsilon) \right\}$, *i.e.* $\int ds' \sin \alpha' \cos \epsilon$, *i.e.* $\int ds' \sin \alpha'(1 - \frac{\epsilon^2}{2} + \ldots)$. Thus we see that this differs from the true value $\int ds' \sin \alpha'$ by $\frac{\epsilon^2}{2}\int ds' \sin \alpha'$, a quantity of the second order. If only one of the above-described operations is performed it is clear that the error is of the first order in ϵ, while by performing both operations and taking the mean, the error is reduced to a quantity of the second order. The construction of Amsler's Polar Planimeter does not permit of performing the double operation. Various forms, however, of Compensation Planimeters have been constructed by the Swiss firm Coradi and the German firm Ott.

Another source of error is the slipping of the integrating wheel. This error, in so far as the inequalities of the surface on which the wheel rolls contribute to it, has been obviated in what are called disc planimeters. In these instruments a circular platform is provided, on which the integrating wheel rests, and the rotation of the platform causes the wheel to revolve. Under this category there are planimeters constructed by Amsler and Coradi.

In all polar planimeters the integrating wheel has a combined motion of rolling and slipping. There are certain curves, called the slip curves, of a polar planimeter which have the property that when the tracing point moves along them, the integrating wheel slips without rolling, and hence the reading of the wheel is constant. The accuracy of the results given by the instrument is increased by arranging as far as possible that the tracing point moves orthogonally to the slip curves. For a discussion of these curves the reader should refer to a paper by A. O. Allan.[1]

(b) *Linear Planimeters*

In these planimeters, as already explained, the curve C′ is a straight line. These instruments consist of a carriage which moves along a rail, the end B of the arm of constant length being fixed to a point of the carriage. As in the case of polar planimeters, devices have been incorporated to avoid the errors to which we have already referred.

(c) *Planimeters of Prytz and Petersen*

If the arm of constant length AB be always tangent to the curve C′, then α' is zero for all positions, and by referring to the general formula it is seen that the elementary area swept out by AB is $\frac{1}{2}l^2 d\phi$, and thus the total area swept

[1] *Phil. Mag.*, p. 643, April 1914.

out is given by $\frac{1}{2}l^2\int d\phi=\frac{1}{2}l^2\phi$, where ϕ is the angle turned through by the arm in making a complete circuit of the curve. The first instrument based on this principle is that due to Prytz of Copenhagen. The area swept out by the arm is made up of the required area S of the curve C and the area (which will be described in the opposite sense) between the curve C′ and the initial and final positions of the arm AB. This latter area can be shown to be

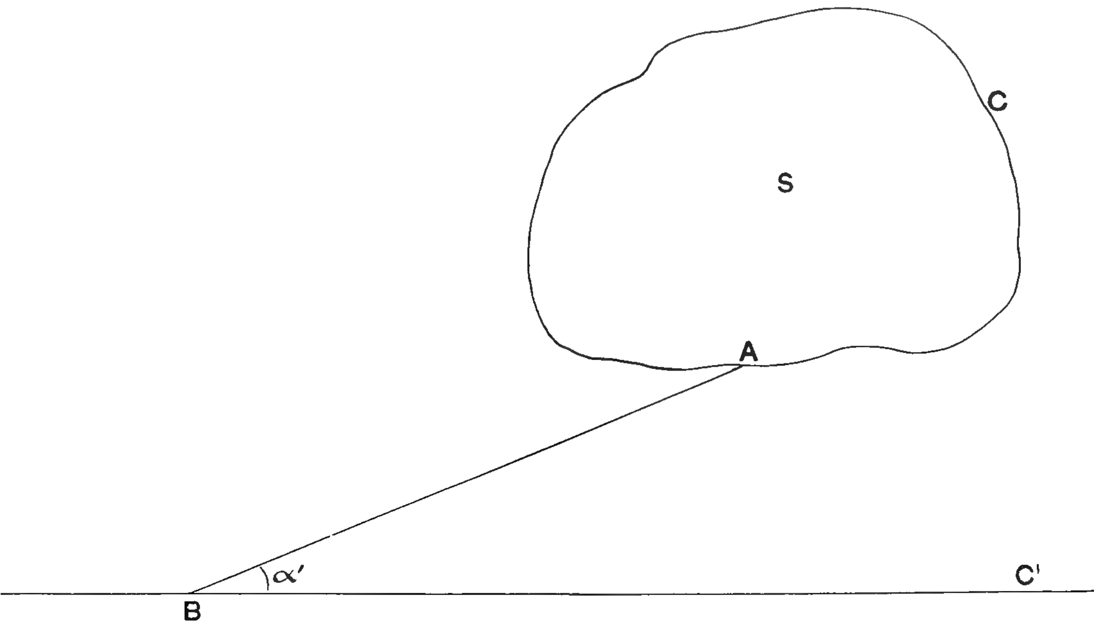


FIG. 6.

approximately $\frac{1}{2}l^2\phi$ if the normal to AB at A, in its initial position, divides the curve into two nearly equal portions.

$$\therefore \quad S-\tfrac{1}{2}l^2\phi=\tfrac{1}{2}l^2\phi$$
$$i.e. \quad S=l^2\phi.$$

Hence, if AB, AB′ be the initial and final positions of the arm, the required area is equal to the product of l and the length of the arc BB′ of the circle whose centre is A and radius l.

Prytz's instrument consists of a metal arm AB, bent at right angles at both ends, as in fig. 7. The end B is in the form of a knife edge, while A is the tracer. It is clear that B can only move freely along the line AB, and thus when A is made to describe the given curve, the point B traces a curve such that AB is always tangent to it. In Prytz's own theory of the instrument he starts the tracer at a point O interior to the area to be measured, moves it along a radius vector, makes a complete circuit of the curve, and returns to the point O by the same radius vector, and he shows that if O be approximately the centre of gravity of the area, the area required is given approximately by $l^2\phi$ as above.

F. W. Hill,[1] in a paper dealing with the Hatchet (Prytz) Planimeter,

[1] *Phil. Mag.*, xxxviii. 265, 1894.

has investigated the theory of the instrument, and develops a formula for the area when O is any interior point of the area; he also deduces limits within which the chord can be measured instead of the arc $l\phi$ if O be near the centre of gravity.

In Goodman's form of Prytz's instrument a part of the arm is bent into a graduated circular arc of radius l. This enables the required area to be got by measuring the arc BB′ by means of this scale, and the scale is calibrated so as to give the reading in units of area. Kriloff [1] has substituted a sharp-edged wheel for the knife edge, and claims greater accuracy in the results obtained.

Another instrument which, like Prytz's, has the peculiarity of not having an integrating wheel, is the planimeter of Petersen. In this the arm of constant length is constrained to move, always keeping parallel to a fixed direction,

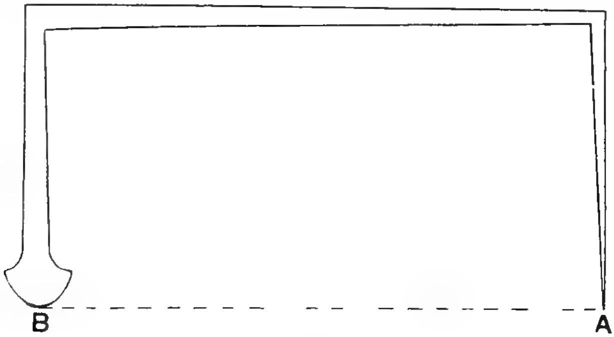


FIG. 7.

and the area swept out by the arm is registered on a linear scale, which is perpendicular to the arm.

MISCELLANEOUS PLANIMETERS

Here we include several instruments which have been designed for special purposes. Among these may be mentioned the spherical planimeter of Amsler, capable of measuring areas on a spherical surface; the stereographic planimeter, also due to Amsler, which measures spherical areas by measuring the corresponding area on the stereographic projection; the mean ordinate planimeters of Durand [2] (giving the mean ordinate of a polar diagram), and that of Schmidt,[3] which gives a high degree of accuracy, but is complicated in construction; Bryan's [4] planimeter, which is useful when a diagram recorded on a drum is of varying scale; the planimeter of Hine-Robertson, in which the slipping of the integrating wheel along an axis measures the area; the Lippencott [5] planimeter, a modification of the above; the interesting instruments of the type called polar coordinate planimeters, which have been devised, but none of which, according to Henrici,[6] have ever been constructed.

[1] *Bull. Acad. Sci. St Petersburg*, xix. 221–227, 1903.
[2] *Amer. Soc. Mech. Eng.*
[3] *Zeitschrift für Instrumentenkunde*, xxv. 261–273, 1905.
[4] *Engineering*, lxxiv. 740, 742–743, 1902.
[5] Greenhill, *Engineer*, lxxxviii. 614–615, 1899.
[6] *B.A. Report*, 1894.

Further information as to details of construction, etc., is given in *Dyck's Catalogue*, Morin's *Les Appareils d'Intégration*, and Henrici's article on Planimeters in the *B.A. Report*, 1894, to all of which we are indebted.

(1) **Exhibit of Planimeters.** By G. Coradi, Zürich

Earliest Form of the Rolling Planimeter

Fig. 8.—Earliest form of the Rolling Planimeter.

This was constructed in 1883.

Rolling-Sphere Planimeter

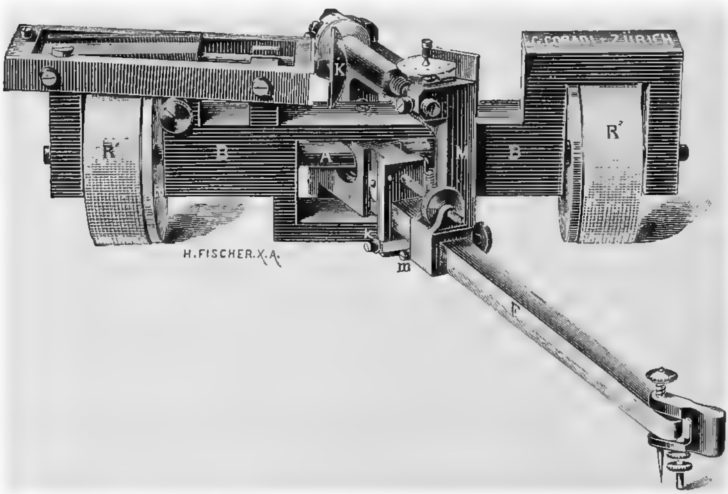


Fig. 9.—Rolling-Sphere Planimeter.

Fig. 9 represents the instrument to about half size.

The guide line of the pivot of the tracer arm F is, as with all linear planimeters, a straight line. The instrument rests on the diagram at three points, the two rollers R′ and the tracer F or its support *s*. In the frame B the axle A works in two centre screws which have their threads in the frame B.

The two cylindrical rollers R′ are rigidly connected with the axle A: they are of equal diameter, coaxal with the axle A, and provided on their circumference with a kind of dotted milling in order to prevent slipping.

On the face of one of these rollers is a wheel with fine teeth. In it gears a small toothed wheel (not shown in the drawing) which is fixed on the steel axle of the spherical segment K. This axle is supported in a horizontal frame. Outside the plate on a cone of this axle, a spherical segment K is fixed, its axis being coincident with that of the axle. The left part of the frame of the axle, and consequently the sphere itself, may be raised somewhat on turning about horizontal pivots engaging with the frame B. It falls by its own weight until the small wheel on the axle rests on the wheel of the cylindrical roller, whereby the proper gearing is secured automatically.

The axle A and the axle of the sphere are parallel and in the same vertical plane.

A spiral spring suspended from the frame M on the one side, and from the tracer arm sleeve on the other, draws the frame M up against the spherical segment *k*, so that the measuring roller is always in contact with the spherical segment.

A screw with a cylindrically milled head, in the frame M, which presses against the tracer arm, enables the frame M to be moved gently away from the sphere, thus destroying the contact between the sphere and cylinder.

The tracer arm can make an angular motion of about 30° to left and right of the base; the magnitude of the movement in the direction of the base is unlimited. This instrument can consequently in one operation measure areas of unlimited length and of a width equal to the length of the tracer arm used.

With the rolling-sphere planimeter the measuring roller performs exclusively *rolling movements* on the surface of an accurately spherical segment; the result of the turning of the roller is therefore unaffected by the slipping or the condition of the paper whereon the figures to be measured are drawn.

ROLLING-DISC PLANIMETER

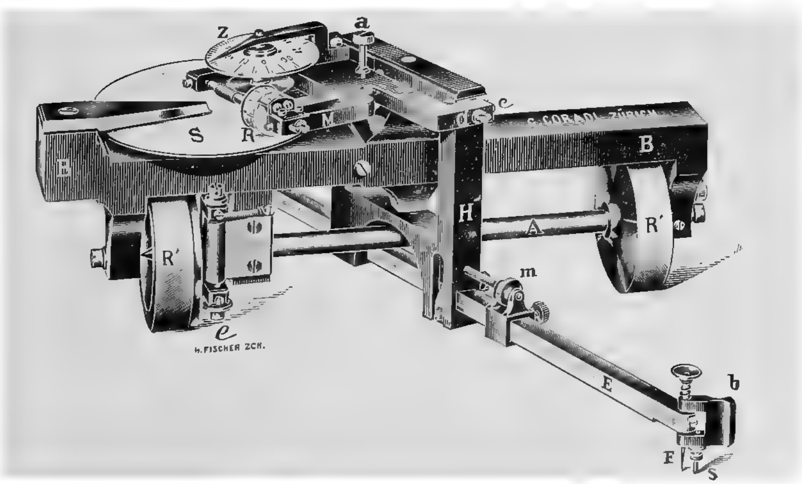


FIG. 10.—Rolling-Disc Planimeter, with gliding movement of the measuring roller.

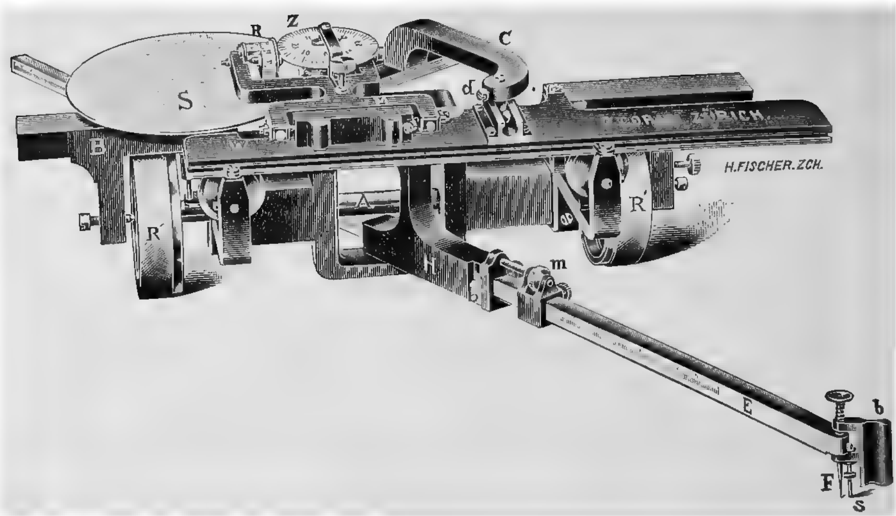


FIG. 11.—Rolling-Disc Planimeter, with pure sine movement of the measuring roller.

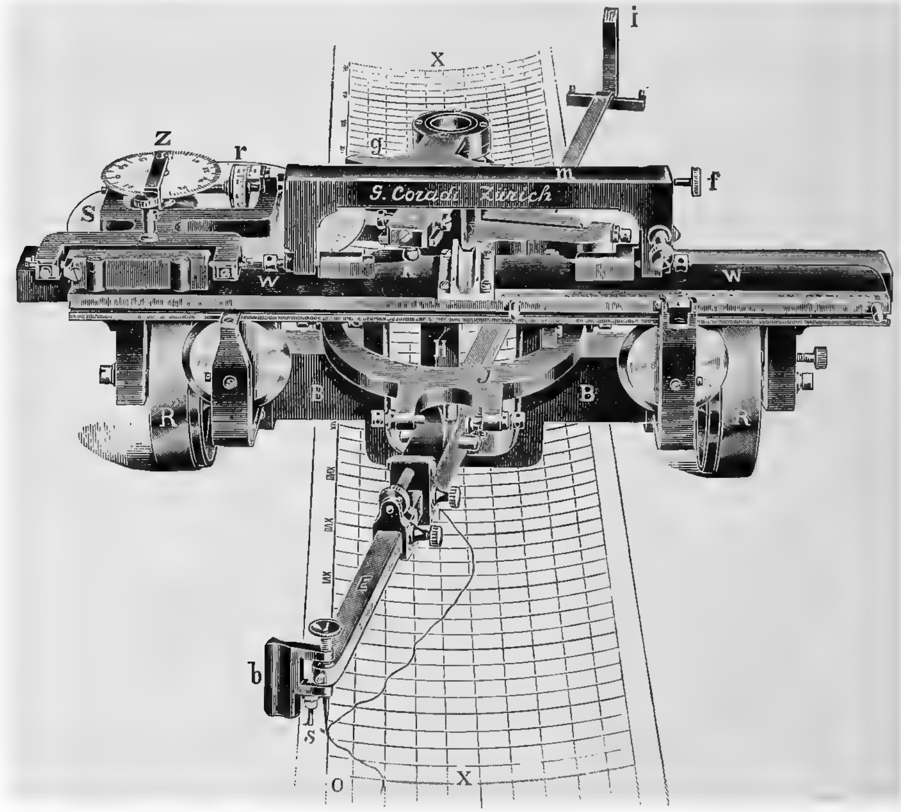


FIG. 12.—Rolling-Disc Planimeter specially arranged for the evaluation of diagrams with curved ordinates.

The displacement of the measuring roller on the disc results from a toothed segment engaging with the rack on the carriage of the measuring roller. This rack may be released and a harmonic lever closed so that the

planimeter acts in the same manner as the rolling-disc planimeter, with pure sine movements of the measuring roller.

(2) **Exhibit of Planimeters.** By A. Ott, Kempten, Bavaria.

1. Compensating Planimeter "Pandero"

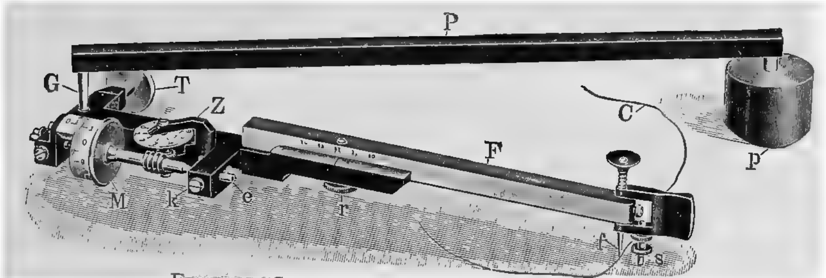


Fig. 13.—Compensating Planimeter "Pandero."

Needle-pole Compensating Planimeter with short, graduated tracer arm, adjustable within a narrow limit; as a rule set for the vernier unit ·1 sq. cm. (By computing figures drawn to a definite scale the area is obtained by multiplying the reading of the roller by the area scale of the drawing.)

2. Compensating Planimeter "Papetos"

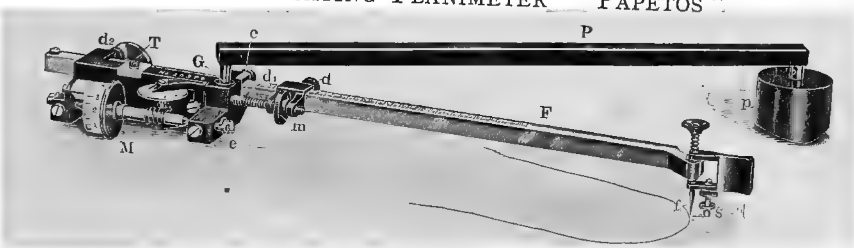


Fig. 14.—Compensating Planimeter "Papetos."

Compensating Planimeter with graduated tracer arm, adjustable to its full length; adjustment by vernier; slow motion for accurate setting to any scale and to allow for shrinkage of paper.

3. Compensating Planimeter "Parapet"

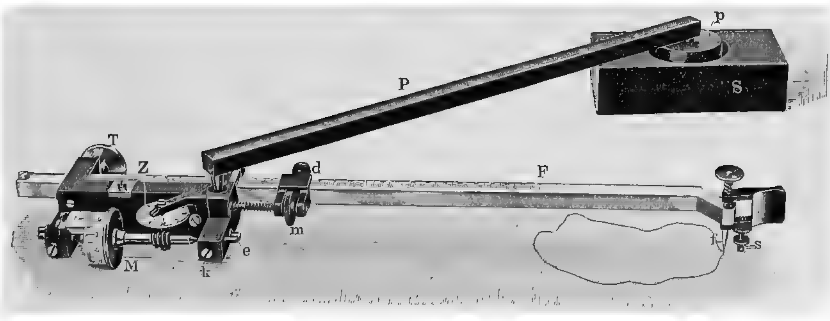


Fig. 15.—Compensating Planimeter "Parapet."

Ball-pole Compensating Planimeter like No. 2, but with device for adjusting the parallelism of the axes by the user.

4. Universal Planimeter and Radial Averaging Instrument

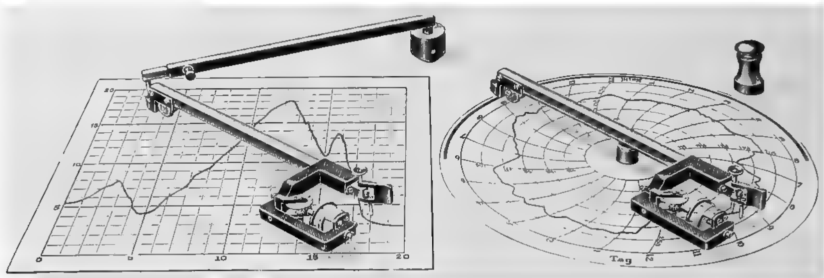


Fig. 16.—Universal Planimeter.

The Universal Planimeter is designed for the computation of areas and the determination of the mean ordinate of diagrams of self-recording apparatus drawn either on strips or on circular charts.

The instrument is essentially a Compensating Planimeter with a fixed tracer arm and a vernier. It has a range of tracing equal to the area of a ring formed by two concentric circles of 5-inch and 29-inch diameter respectively. In using it as a radial averaging instrument, the range of tracing is equal to a ring formed by two circles of 1 inch and 13 inches respectively.

By connecting the tracing arm with a heavy brass roller the instrument can further be used as a Rolling Planimeter.

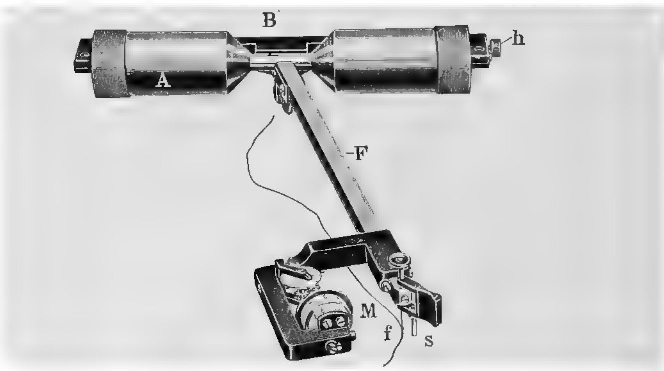


Fig. 17.—Universal Planimeter with Roller.

Roller for the Universal Planimeter (*i.e.* the part AB of above illustration).—The roller may be connected very conveniently with the tracing frame, allowing areas of any length and a width of 11 inches to be measured with the pattern "Paregol," and of 22 cm. with the pattern "Pardune."

Parts of the Universal Planimeter

The illustration (fig. 18) shows the single parts of the Universal Planimeter. According as the tracing frame TFM is connected with the pole arm GP*p* or the roller ABC, or again with the centre D, we have a Polar Planimeter,

or a Rolling Planimeter, or a Radial Averaging Instrument, each of which possesses some characteristic advantage.

The use of the instrument as a Polar Planimeter is exactly the same as in the case of Ott's Compensating Planimeter of the simple pattern, the only difference being in the value of the vernier unit. In measurements with the pole inside the figure, the constant is zero, and therefore need not be taken into account.

This will easily be understood when considering that the constant is equal to the area of a circle described by the tracing point about the pole. Now the plane of the measuring disc is radial. As with the Universal Planimeter, the pole arm and the tracer arm are of the same length, and as this again is equal to the distance of the plane of the roller from the joint *g*, the area of the said circle diminishes to a point.

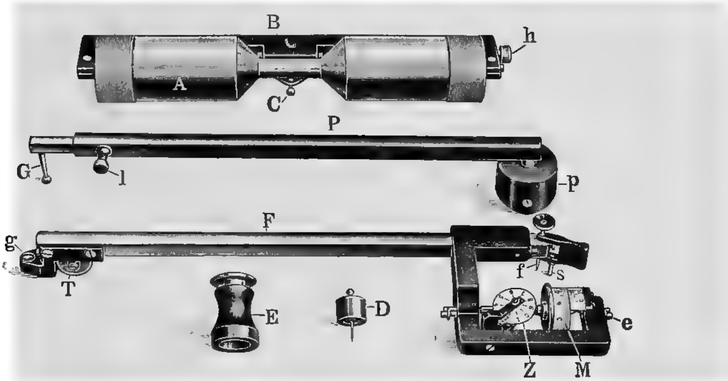


FIG. 18.—Single Parts of Universal Planimeter.

In this instrument, too, the roller is kept conveniently under observation, thus avoiding any errors that might otherwise be caused by unnoticed hindrances to the smooth turning of the roller, such as the edges of the sheet or particles of rubber. The accuracy of this planimeter is, if anything, even higher than that of the ordinary Compensating Planimeter.

The Radial Averaging Instrument may be used in the computing of diagrams with straight or curved ordinates as long as they have equidistant intervals. The method of using the instrument is the following :—

By aid of the punch E press into the drawing-board the centre D. Under this head is placed the diagram. Then set the tracing frame on the chart and insert the ball-shaped head of the centre pin into the groove at the lower side of the tracing arm. In this manner the arm is securely guided with regard to the centre of the diagram. Now set the tracing pin to the point of commencement of the registered curve, and trace the whole curve from left to right. Then follow the radius to or from the centre to the same distance as that of the starting-point, and again take the reading of the roller. The difference of the readings multiplied by 0·0004 denotes in inches the mean radius of the diagram, if the registration is exactly one round of the chart. If it is less or greater, the obtained reading of the roller must be reduced to one revolution. If, for instance, the time of registration is only sixteen hours, then the reading must be multiplied by 24/16. To obtain

the final result the radius of the base circle must be subtracted from the mean radius of the diagram thus obtained.

The Radial Averaging Instrument possesses a few specially interesting features, not only from a practical but also from a theoretical point of view. If we denote by r and ϕ the polar co-ordinates of the curve traced, then the measure of turning, θ, of the circumference of the integrating roller is defined by the equations

$$\theta = \int_{\phi_1}^{\phi_2} r d\phi \quad \text{or} \quad \theta = \int_{r_1}^{r_2} r \frac{d\phi}{dr} dr.$$

With certain algebraic curves, such as straight lines, circles, ellipses, etc., the above integrals lead to hyperbolic, cyclometric, and elliptic functions which, with the aid of our instrument, may be determined in a purely mechanical way.

The relation between the turning of the roller and the *area* defined by the tracing of the pin can be derived in the following manner. We have

$$\theta = \int r d\phi = \iint dr d\phi = \iint \frac{r dr d\phi}{r} = \int \frac{df}{r}.$$

In consequence thereof θ is equal to the sum of quotients of the single elements of area df by their relative distances r from the centre of measurement, or, put differently, θ is equal to the potential of the area enclosed by the curve (the density of mass supposed to be unity) about this centre.

References.—A. Amsler, "Mechanische Bestimmung des Potentials und der Anziehung," *Carl's Repertorium für experimentelle Physik*, xv. S. 399, 1879; Derselbe, "Ueber mechanische Integrationen," im *Katalog math. und math.-phys. Modelle, Apparate und Instrumente*, herausgegeben von W. Dyck, München, 1892; A. Russel and H. H. Powles, "A New Integrator," *The Engineer*, 1896; T. H. Blakeley bezw. Dr Rothe, "Ueber eine Methode zur mech. Auswertung der hyperbolisch-trigonometrischen Funktionen," *Zeitschrift für Instrumentenkunde*, xxiv. S. 151, 1904.

(3) **Exhibit of Planimeters.** From the Department of Natural Philosophy, University of Edinburgh

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IV. The Use of Mechanical Integrating Machines in Naval Architecture. By A. M. Robb, B.Sc.

General Description of Machines in Use

The mechanical integrating machines used in naval architecture may be divided into three main classes:—(i) Planimeters, measuring areas; (ii) Integrators, measuring areas, and first and second moments of these areas about chosen axes; (iii) Integraphs, tracing directly the integral curve of any curve round which the machine is guided. The first two classes of machine are absolutely essential to the naval architect, and are in daily use

in all great shipyards. The last class of machine is not in common use. There are probably only three or four in this country.

Planimeters.—There are several types of planimeter; but only one type—the polar planimeter—is in common use. The first polar planimeter was put on the market in about 1854 by Amsler, and his machine of the present day is practically the same as the original one. An illustration of an Amsler planimeter is given in fig. 1, and a diagram indicating the manner in which it is used is given in fig. 2.

The pole O is kept in a fixed position, relatively to the area to be measured, by a small weight. The outer end of the pole arm OA describes a circular

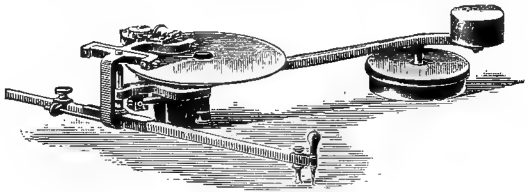

FIG. 1.

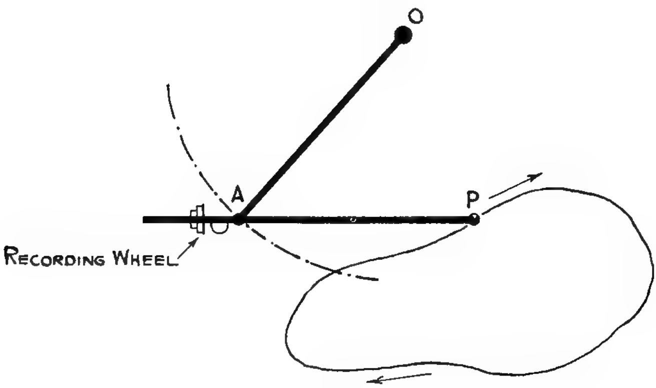

FIG. 2.

arc about O, while the tracing point P is being guided round the area. The reading of the wheels on the tracer arm AP is noted before starting, and again after the tracing point has been guided completely round the boundary of the area. The difference between the two readings is a measure of the given area. The constant by which the difference of the readings must be multiplied depends on the circumference of the recording wheel and on the distance between A and P. The motion of the recording wheel on the paper is partly one of rotation about its axis, and partly one of translation parallel to its axis. The latter motion has no effect on the reading. Hence, if the tracing point be guided round such an area that the recording wheel is always moving parallel to its axis, there will be no reading. This is the case when the pole O is at the centre of the circle, whose diameter is such that the plane containing the edge of the recording wheel passes through O when the tracing point is on the circumference (see fig. 3).

This circle is known as the base circle, and its area is constant for each type of polar planimeter.

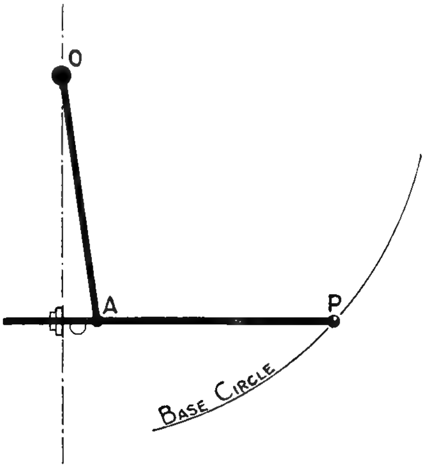

FIG. 3.

With the ordinary polar planimeter there is a possible source of error, due to the axis of the recording wheel not being parallel to the tracer arm. In order to eliminate this error, if present, it is necessary to take the mean

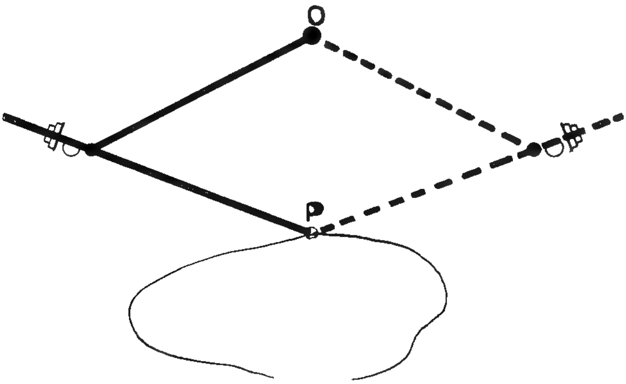

FIG. 4.

of two readings from an area, one with the pole to the left of the tracer arm, the other with the pole to the right, as indicated in fig. 4. With the Amsler planimeter this cannot be done. The tracer arm is mounted *above* the pole arm, and so the range of the tracing point is restricted. The Coradi Compensation Planimeter, illustrated in fig. 5, allows this double reading to be

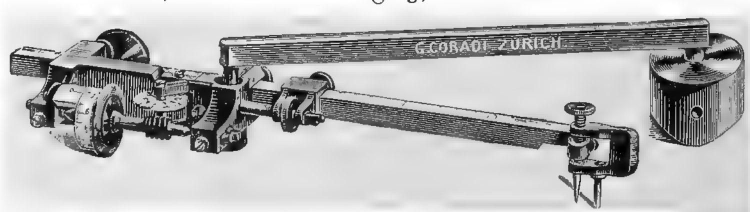

FIG. 5.

made, and so any error due to improper mounting of the recording wheel can be eliminated.

Another type of Amsler planimeter is illustrated in fig. 6. The advantage of this type lies in the fact that the recording wheel works on a revolving disc

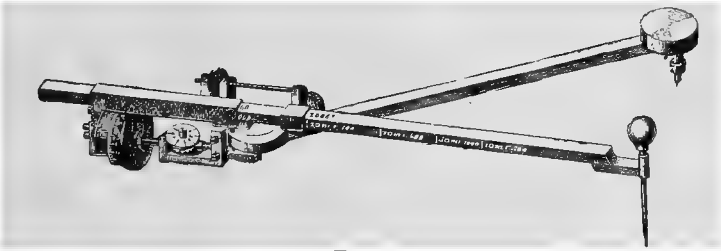

Fig. 6.

instead of on the surface of the drawing, thus ensuring greater accuracy when measuring drawings which have been creased through folding.

A modification of the polar planimeter is illustrated diagrammatically in fig. 7.

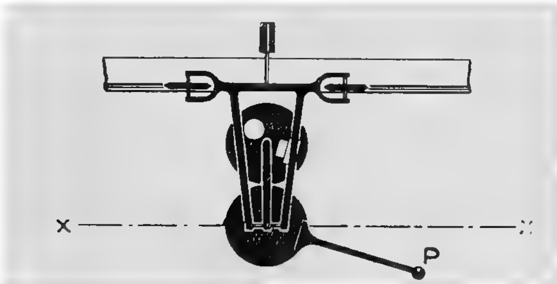

Fig. 7.

The instrument is constrained to move in a straight line by a guide bar. The axis XX corresponds to the circular path of the outer end of the pole arm in a polar planimeter. In effect, this machine is equivalent to a polar planimeter with an infinitely long radial arm.

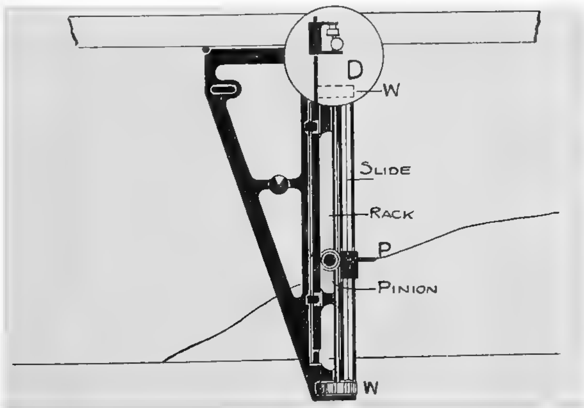

Fig. 8.

A modern planimeter whose working principle is the same as that of the earliest forms is illustrated diagrammatically in fig. 8.

This machine was designed by Mr W. R. Whiting. It is guided along a straight edge, two small vertical rollers being fitted at the corners of the frame. The recording wheel works on a disc D, which is rotated by the wheels W on which the machine travels. The tracing point is free to slide along

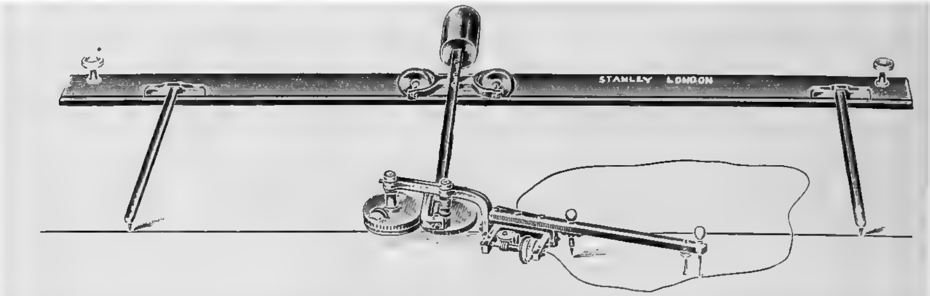

FIG. 9.

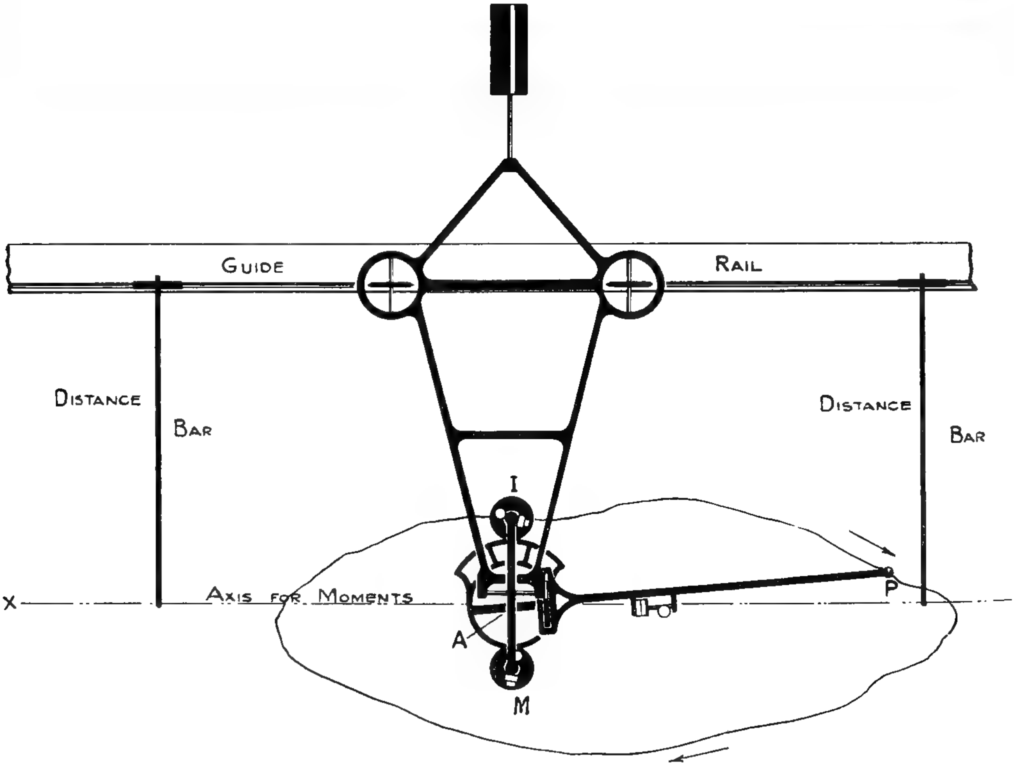

FIG. 10.

the main framework, and is connected by a rack and pinion to the bar on which the recording wheels are mounted. As the tracing point is moved outwards the recording wheel moves outwards on the disc and so is rotated more quickly. That is, for uniform linear motion of the machine the rate of rotation of the recording wheel depends on the ordinate of the curve round which the tracing point is being guided.

Integrators.—The Amsler integrator is, practically speaking, an extension

of the linear planimeter illustrated in fig. 7. The smaller sizes measure areas and moments; the larger sizes measure areas and first and second moments. An illustration of a small Amsler integrator is given in fig. 9, and a diagram of a large one in fig. 10.

The wheels mounted on the bar AP record the area, those mounted in the disc M record the moment of the area about the axis XX, and those in the disc I record the moment of inertia about XX. The machine runs along a guide rail which is set parallel to the chosen axis by the distance bars indicated. With the Amsler integrator it is necessary to trace completely round a figure whose area and moment are required. Hence, if it is required to measure areas and moments up to a series of stations along any area, it is necessary to trace completely round each individual portion.

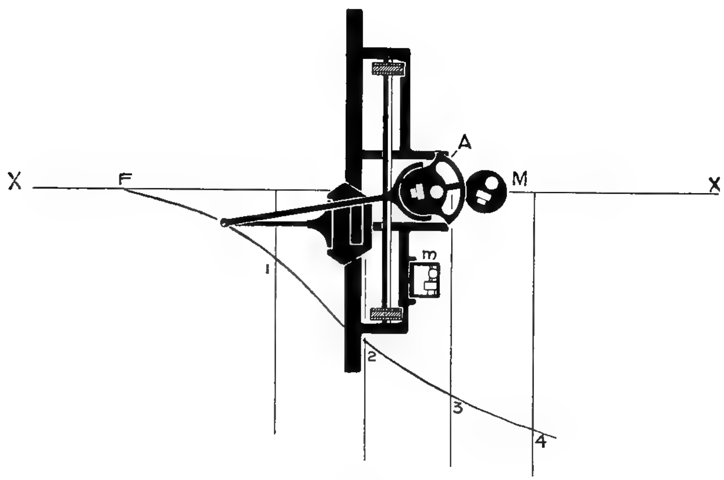

Fig. 11.

An improved form of integrator which does away with the necessity for tracing completely round a figure has been invented by Mr J. G. Johnstone, B.Sc. A diagram of this machine is given in fig. 11.

The machine is set to travel along a chosen axis, parallel motion being ensured by two non-slipping wheels. The wheel A records areas, the wheel M, in conjunction with a wheel *m* recording the advance of the machine, records moments about the axis XX.

In order to measure areas and moments of the figure indicated above to a series of stations 1, 2, 3, 4, perpendicular to the axis, it is only necessary to guide the tracing point from F round the curved boundary. The readings at the points 1, 2, etc., give the areas and moments of the figure up to the respective stations.

The Integraph.—A diagram of the most modern type of integraph, invented by M. Abdank Abakanovicz, and manufactured by Coradi, is given in fig. 12.

The machine is set to travel along a chosen axis, generally the base line of the curve to be integrated. Once it has been set, the non-slipping wheels W are sufficient to ensure that the motion is along the axis. To facilitate setting, the scale bar carrying the tracing point can be locked centrally on

the main frame. The motion of the recording pen is always parallel to the plane of a small, sharp-edged, non-slipping wheel *w*. By means of the parallel framework shown, the plane of the wheel *w* is maintained parallel to the radial bar. When the tracing point is being guided round the given curve, the scale bar is travelling out along one side of the main frame. Consequently the angle θ between the radial bar and the axis is constantly changing, and so also is the angle θ between the plane of the wheel *w* and the axis.

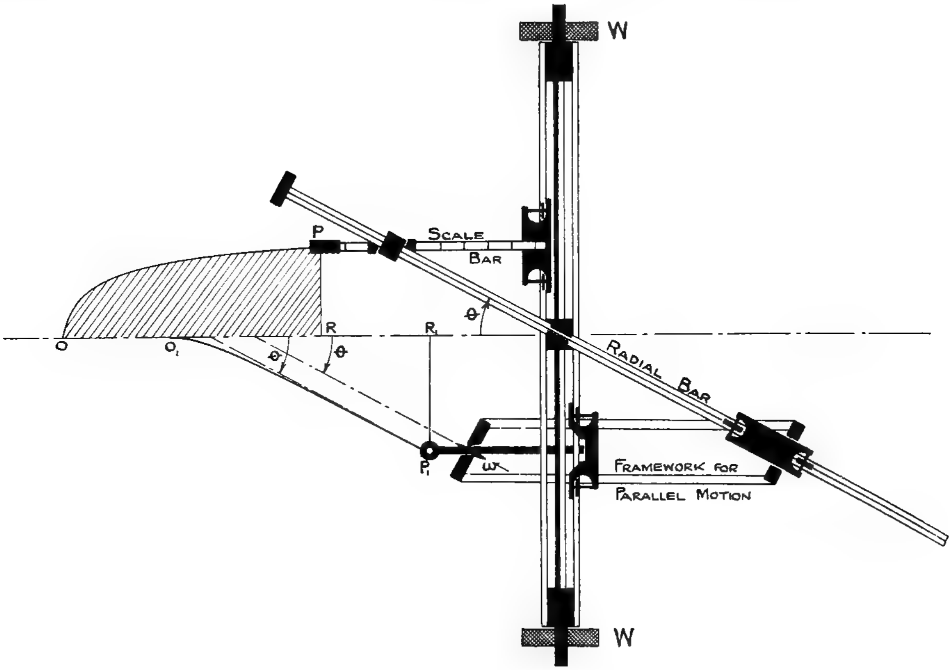

Fig. 12.

Since the wheel *w* is at any instant moving in the direction of its plane at that instant, guiding the tracing point round a curve such as OP results in the machine drawing a curve such as O_1P_1. The end ordinate R_1P_1 of the curve traced out by the machine is a measure of the area ORP round whose curved boundary the tracing point has been guided.

Methods of Employing Integrating Machines

For practically all measurements of area polar planimeters are used. As a rule the areas to be measured are of such a size that the pole can be kept outside the boundary of the figure, thus avoiding the necessity of making a correction for the area of the base circle. The common method of calculating volumes is to measure cross-sectional areas at definite intervals and integrate longitudinally by Simpson's or Tchebycheff's rules.

When calculating the position of the centre of gravity of a solid, for example, the centre of buoyancy of a ship, it is necessary to employ an integrator. Fig. 13 indicates the arrangement of an Amsler integrator when calculating the vertical position of the centre of buoyancy.

The machine is set to a convenient axis, and for each of a definite series of cross-sections the area and moment are measured. These areas and moments are then integrated by one of the arithmetical rules in common use, and the immersed volume and the position of its centre of buoyancy above or below the axis determined.

The calculation of stability can be carried out entirely by means of Simpson's or Tchebycheff's rules, but the most common method is to use an integrator to obtain the areas and moments, about a chosen axis, of a series of cross-sections spaced to suit one of the above-mentioned arithmetical rules. The arrangement of an Amsler integrator when calculating stability is indicated in fig. 14.

The plan giving cross-sections of the ship at definite intervals is known as the "body plan." On this plan a series of radial lines, at about 15 or 20 degrees interval, is drawn through a chosen point on the centre line, referred

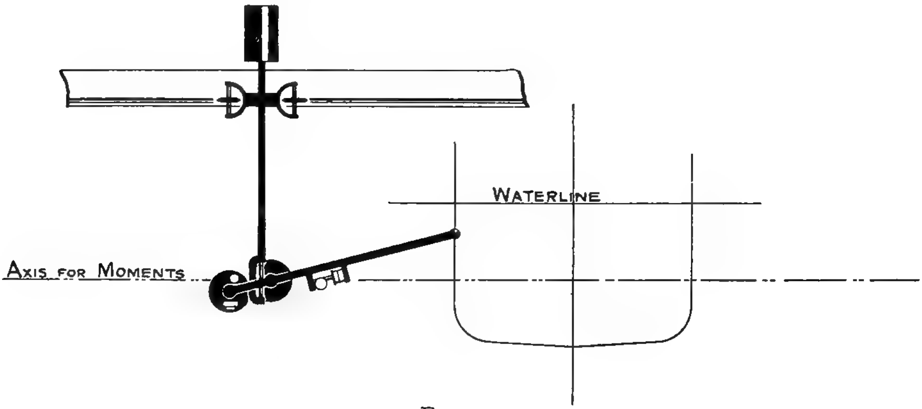

FIG. 13.

to as the "assumed C.G." The assumed C.G. is for convenience generally chosen near the actual centre of gravity of the ship. The integrator is then set so that the axis coincides with one of the radial lines. For this position of the machine the area and moment of each section are then measured up to each of a series of lines, four or five in number, drawn perpendicular to the axis. These lines, marked WL in fig. 14, represent different immersions. This operation is performed for each radial line in turn from 0 degrees to 90 degrees, and occasionally to 180 degrees. Then by longitudinal integration by the suitable arithmetical rule are obtained for each inclination four or five values of the immersed volume, or displacement, and the corresponding moments of these displacements, about the axis. Since the axis of the machine corresponds to the vertical, the moments of displacement, divided by the corresponding displacements, give the distances from the vertical through the assumed C.G. of the line of action of the buoyancy at each inclination for the different immersions. The stability is measured by the arm of the couple formed by the upward force of buoyancy and the weight of the vessel acting downwards through the centre of gravity. It is not possible to set the integrator with the axis passing through the actual centre of gravity, as this is a variable point depending on conditions of loading.

As a rule the stability is calculated for a number of different conditions of loading. The correction for the stability arm consequent upon the axis

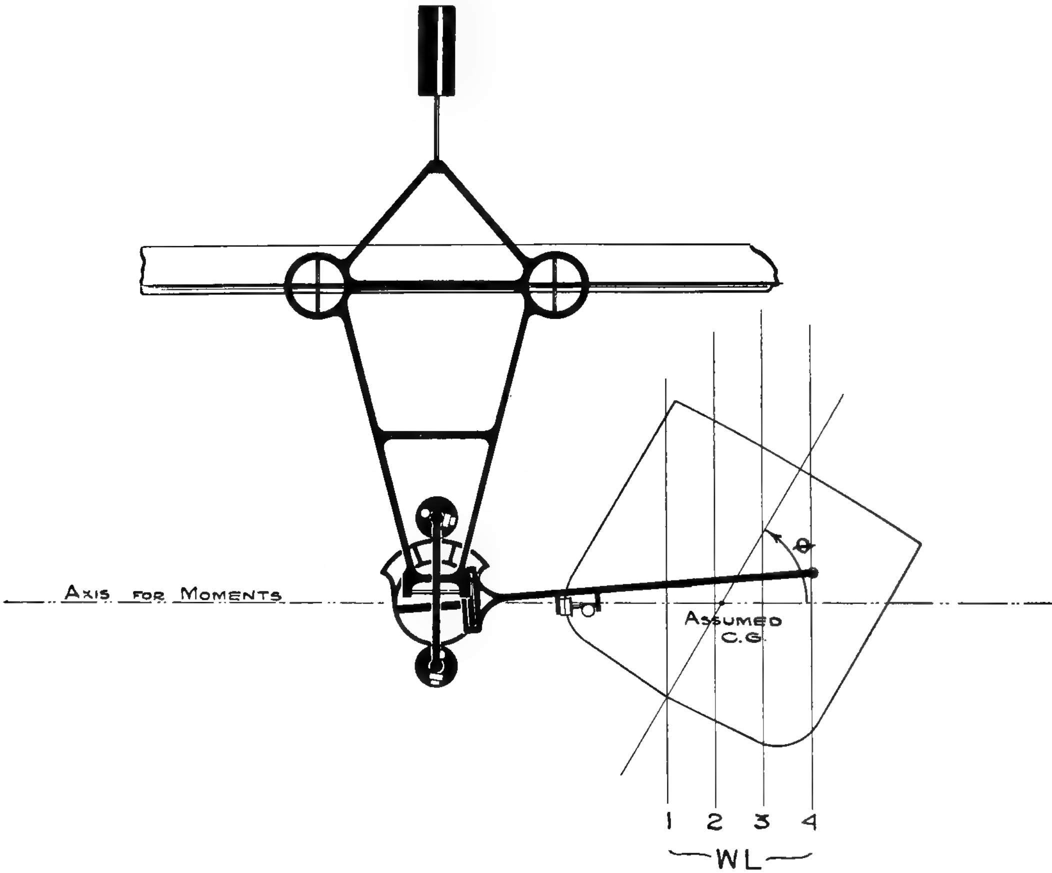

FIG. 14.

not passing through the centre of gravity is very easily made. Having obtained a series of values of the stability arm for a given displacement and condition of loading, they are plotted on a base of angle of inclination. A typical " stability curve " is indicated in fig. 15.

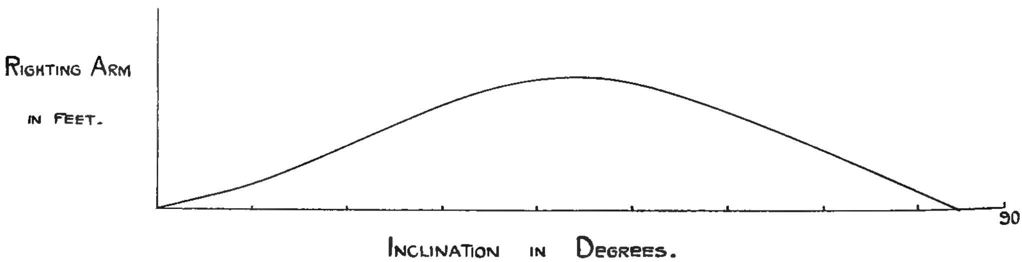

FIG. 15.

It will be seen on reference to fig. 14 that with the Amsler integrator it is necessary to trace round each cross-section up to WL 1, and take the

readings; then trace round the portion between WL 1 and 2, take the readings, and add them to those up to WL 1; and so on for the other portions. With the form of integrator invented by Mr Johnstone the procedure is to trace round the outside of the section up to WL 1, take the readings, continue to 2, 3, and 4, taking readings at each WL. Then transfer the tracing point from the outside of WL 4 to the inside. This does not alter any of the readings. From the inside of WL 4 the pointer is traced round the section, readings being taken at 3, 2, 1, and axis. The area readings for the inside boundary are then added to the corresponding ones for the outside boundary; the moment readings for the inside are subtracted from the corresponding ones for the outside. This method simplifies considerably the work of obtaining a series of values of areas and moments for a cross-section.

The use of an integrator to determine a moment of inertia is very uncommon. In order to use it for the determination of the metacentre, it is

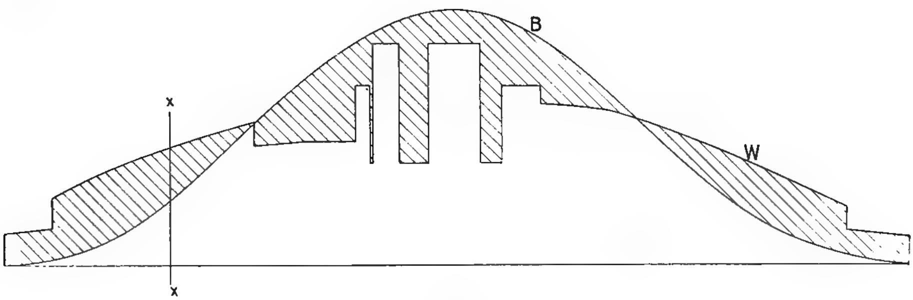

Fig. 16.

necessary to have a series of longitudinal sections parallel to the water-line of the vessel. These are not available for the preliminary calculation in the early stages of a design. For the finished ship these planes can easily be obtained, but other calculations are required which make it more convenient to employ arithmetical rules for the determination of the moment of inertia.

The integrator is also employed in strength calculations. In these cases only areas are required, but the figures are as a rule beyond the scope of a planimeter.

In fig. 16 the curve B represents the distribution of buoyancy, or support, along a ship. Any ordinate of this curve is proportional to the cross-section of the immersed portion of the ship at the corresponding point.

The curve W represents the distribution of the weight. The areas under these curves must, of course, be equal, since they represent the total buoyancy and total weight. Any vertical intercept between these two curves represents the unbalanced buoyancy or weight at the corresponding position in the ship. The sum of all the elementary unbalanced forces on one side of any section is the shearing force at that section. Hence the area of the shaded portion to the left of XX in fig. 16 is a measure of the shearing force at XX. So that in order to obtain a curve showing the variation in shearing force along the ship, it is necessary to measure up to a series of stations the

areas enclosed between the weight and buoyancy curves. Area above the buoyancy curve may be reckoned positive; area below may be reckoned negative. Since the total unbalanced load is zero, the total shaded area is zero. That is, the shearing-force curve meets the base line at the ends.

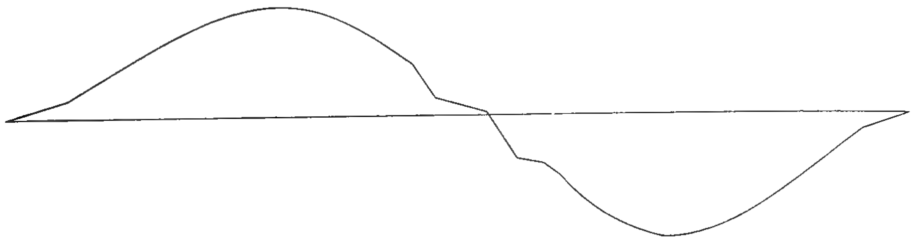

FIG. 17.

A typical shearing-force curve is given in fig. 17.

The area enclosed under a shearing-force curve on one side of any chosen section is a measure of the bending moment at the corresponding section in the ship. Consequently, in order to obtain a curve of bending moments

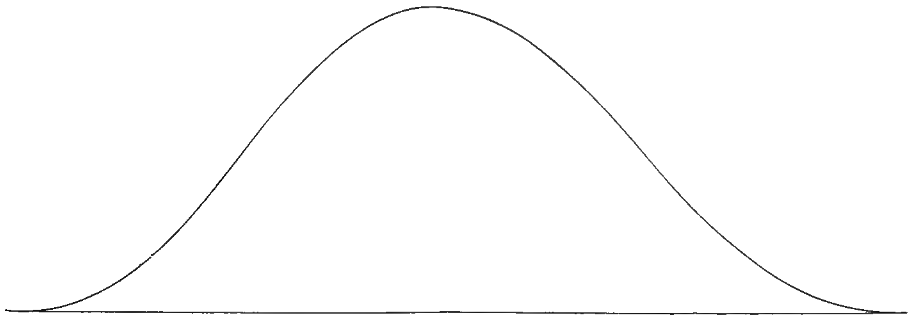

FIG. 18.

it is necessary to measure the areas under the shearing-force curve up to a series of chosen stations. These areas are then plotted on a base representing length, and the resulting curve of bending moments is generally of the form indicated in fig. 18.

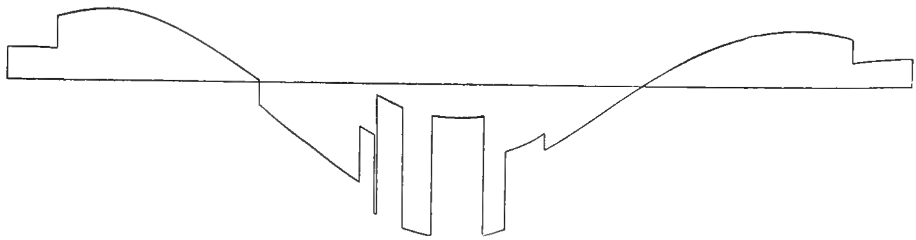

FIG. 19.

The determination of shearing-force and bending-moment curves is very much simplified by the use of the integraph. It is convenient in this case to employ a curve of loads (see fig. 19).

Any ordinate of this curve represents the difference between the weight and buoyancy over one foot of the length of the ship at the corresponding

point. This curve is simply the shaded portion of fig. 16 transferred down to a straight base line. The first integral curve of this load curve is the shearing-force curve for the vessel. Hence, if the integraph be set with its axis along the base line and the pointer be traced round the curve, the machine will draw the shearing-force curve. In the same way, if the axis of the integraph be set along the base line of the shearing-force curve and the pointer be traced round the curve, the machine will draw the bending-moment curve.

For detailed discussions of the theory of the machines herein described, reference may be made to any of the following :—

Les Appareils d'Intégration, H. de Morin.

Report on Planimeters, Professor O. Henrici, British Association, 1894.

Mechanical Integrators, Professor H. S. Hele-Shaw, Institution of Civil Engineers, 1884–5.

The Application of the Integraph to some Ship Calculations, J. G. Johnstone, Institution of Naval Architects, 1907.

An Improved Form of Integrator, J. G. Johnstone, Institution of Engineers and Shipbuilders in Scotland, 1913–14.

List of Instruments on Exhibition

Amsler planimeter.
Amsler planimeter for very large or very small areas.
Amsler revolving disc planimeter.
Coradi compensation planimeter.
Whiting planimeter.
Amsler integrator, small size.
Amsler integrator, large size.
Coradi integraph, latest pattern.
Coradi integraph, earlier pattern.

For these exhibits the author's thanks are due to Glasgow University, and to Messrs W. F. Stanley & Co., Ltd. Messrs Stanley have also lent the blocks from which the illustrations of the machines in the above article have been taken. They have taken a great interest in the exhibition and have freely given their assistance.

V. A Differentiating Machine. By J. Erskine Murray, D.Sc.

(Reprinted from the *Proceedings of the Royal Society of Edinburgh*, May 1904.)

The construction of the differentiator depends on the well-known fact that if the values of a variable quantity be represented on a diagram by the ordinates of a curve, its rate of change, at any point of the curve, is measured by the slope of the tangent at that point.

The machine, then, is guided by hand, so that one line on it remains tangent to the curve, while a tracing point describes on a second sheet of paper a curve whose ordinates are proportional to the slope of the tangent.

Thus, if $y=f(x)$ be the equation to the original curve, the derived curve will have for ordinates the corresponding values of $d(f(x))/dx$. The abscissæ are the same on both curves.

In order that a line may be tangent to a curve it is necessary that two consecutive points on each should coincide. In practice, two black dots on a piece of transparent celluloid are used, the distance between them being about 2 mm.

The plan of the machine is shown in fig. 1. It consists of three parts. Firstly, the large drawing-board ABCD, on which the original curve is placed. Fixed to each long side of this board is a metal rail, one, CE, having

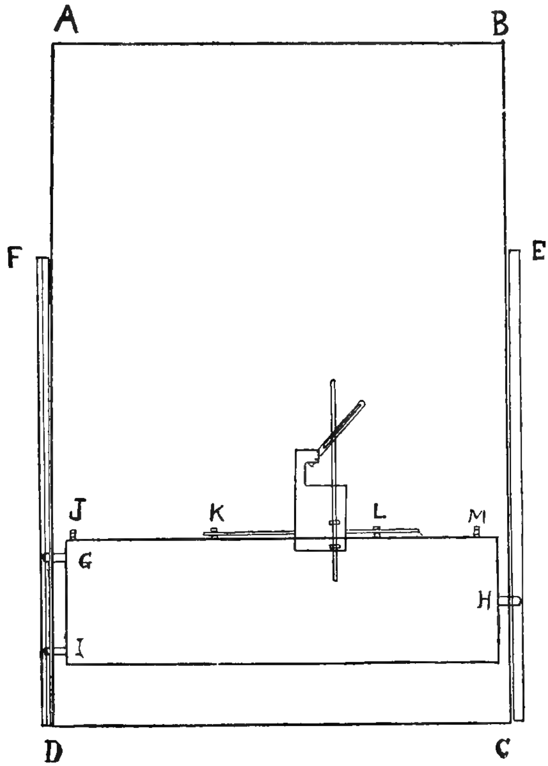

Fig. 1.

a plain surface, and the other, DF, a longitudinal groove of V-shaped section. The second part is a smaller board, GHI, having three spherical feet, two of which run in the groove and the third on the plane rail. This arrangement permits free motion of the smaller board in the direction of the length of the larger one, *i.e.* parallel to the Y co-ordinate. The small board carries the paper on which the derived curve is traced by the machine. Attached to its edge are guides, JKLM, which hold the principal part of the mechanism, allowing it free motion in a right and left line.

This part, shown in fig. 2, consists of a frame ABCD, at one corner of which is a pin, A, which serves as the vertical axis about which the rod PQ revolves in a horizontal plane. PQ has a slot in it, through which passes the pin R fixed to the rod ST. ST is controlled by guides E and F, so that it can only move in a direction parallel to OY.

Below the arm PQ, and fixed rigidly to it below A, is a small plate of celluloid, not shown in the diagram, on the under side of which are two dots by which the machine is guided along the curve. The line through the dots is parallel to PQ. The celluloid rests on the paper on which the original curve is drawn, thus supporting the outer end of the frame ABCD.

Since the distance AV between the pin and the centre line of ST is constant, and since $RV/AV \doteqdot dy/dx$, it is clear that the distance RV through which R is displaced above or below the zero line AV measures the tangent of the angle of slope of the curve, *i.e.* dy/dx. A pen at the end T of ST records

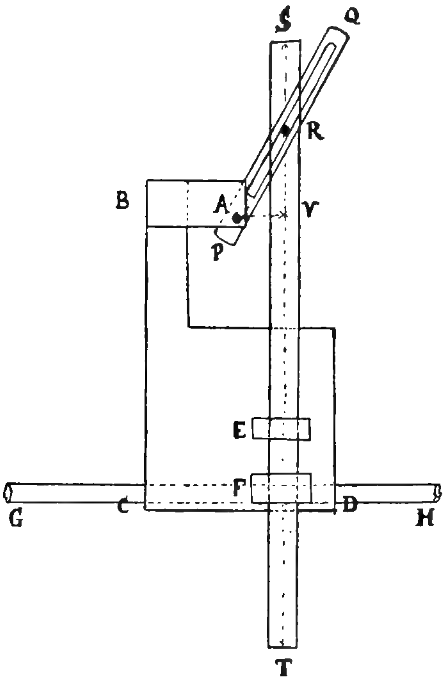

FIG 2.

the movements of R, and therefore traces a curve of which the ordinates are proportional to the rate of change of the ordinate of the original curve. It should be noticed that the purpose of the second board is to eliminate the Y co-ordinate of the original curve. In using the machine the arm PQ is moved so that it remains tangent to the original curve, while the frame ABCD is moved from left to right, and it and the smaller board to and fro as may be necessary in following the curve.

The machine shown has been constructed to deal with curves in which the tangent of the angle of slope does not exceed 5; this is sufficient for almost all experimental or observational results, since it is always possible to flatten out the curve by making the horizontal scale large in proportion to the vertical.

It is, of course, easy to obtain the higher derivatives of the original curve by a simple repetition of the process on the successive curves.

VI. Harmonic Analysis. By G. A. Carse, D.Sc., and J. Urquhart, M.A.

By Fourier's theorem we know that any periodic function $y=f(\theta)$ can be expanded in a series of the form

$$\begin{aligned} y = a_0 + a_1 \cos\theta + a_2 \cos 2\theta + \ \ldots \\ + b_1 \sin\theta + b_2 \sin 2\theta + \ \ldots \end{aligned}$$

where $a_0, a_1, a_2, \ldots \ b_1, b_2, \ldots$ are constants.

The function y may be a known function of θ, or it may be given in the form of a curve got from observations; and in experimental science the latter is the important case.

We have then a given curve, and it is required to determine its equation as a Fourier series, it being postulated that it represents a periodic function whose period may be either assumed or definitely known. Such a problem occurs in the study of alternating currents, sound, heat, terrestrial magnetism, atmospheric electricity and meteorology generally, and hence the necessity for a convenient mode of determining the coefficients in a Fourier series representing a given curve.

Cauchy has shown that analytically the coefficients are given by

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)d\theta$$

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos\theta d\theta \qquad b_1 = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin\theta d\theta$$

$$\begin{matrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \qquad & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \qquad & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \qquad & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{matrix}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta d\theta \qquad b_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta d\theta$$

If y be a known function of θ, then the calculation of the coefficients is a problem in the integral calculus; but, as we have already pointed out, the important case in practice is that in which y is not known explicitly as a function of θ.

In the latter case the methods that have been devised for the solution of the problem can be conveniently divided into—

1. Mechanical methods—by machines called Harmonic Analysers.
2. Arithmetical methods.
3. Graphical methods.

I. Harmonic Analysers

It will be observed that a_0 is the mean ordinate of the curve, and therefore can be determined by an ordinary planimeter, but the other coefficients cannot be determined directly by this means. For the determination of these coefficients harmonic analysers have been invented. The first instrument of

this kind is that due to Lord Kelvin.[1] The basis of this instrument is the disc-sphere-cylinder planimeter of J. Thomson.[2] His mechanism consists of a disc, sphere, and circular cylinder arranged as follows. The disc is capable of rotation about an axis perpendicular to its plane and passing through its centre. The cylinder, which is not in contact with the disc, can rotate about its axis, which is parallel to the plane of the disc. The sphere is always in contact with the disc and the cylinder, and can roll along, keeping in contact with both and not making either rotate. The path of the point of contact with the disc is a diameter of the disc, while the path of the point of contact with the cylinder is a generator, and this diameter is parallel to the generator.

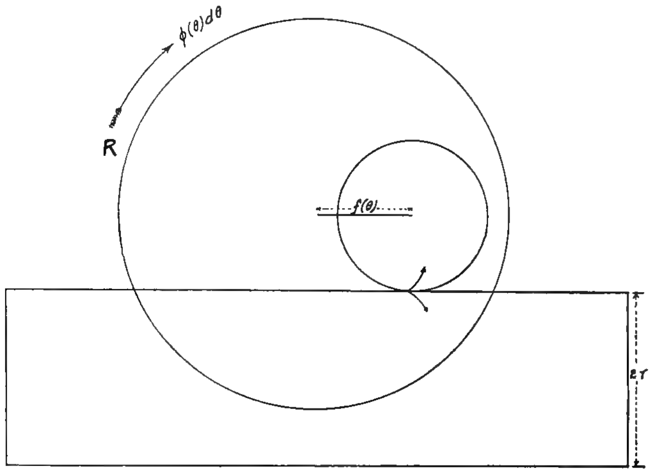

FIG. 1.

As a planimeter, the instrument evaluates an integral of the form $\int f(\theta)d\theta$, but Lord Kelvin[3] pointed out that it can be utilised to evaluate an integral of the form $\int f(\theta)\phi(\theta)d\theta$ in the following manner:—Suppose a point in the circumference of the disc whose radius is R receives an elementary displacement $\phi(\theta)d\theta$, and if the distance between the centre of the disc and its point of contact with the sphere be $f(\theta)$, the point of contact of the sphere with the disc, and therefore the point of contact of the sphere with the cylinder, will move through a distance $\frac{f(\theta)\phi(\theta)d\theta}{R}$; hence if the radius of the cylinder, which may be called the integrating cylinder, be r, the elementary angle dw turned through by the cylinder is $\frac{f(\theta)\phi(\theta)d\theta}{Rr}$, and the total angle w is

$$\frac{1}{Rr}\int_0^\theta f(\theta)\phi(\theta)d\theta.$$

[1] *Proc. Roy. Soc.*, xxvii. 371, 1878. Kelvin and Tait's *Natural Philosophy*, App. B[1]. VII., 1896.
[2] *Ibid.*, xxiv. 262, 1876.
[3] *Ibid.*, xxiv. 266, 1876.

If $\phi(\theta)$ is $\begin{matrix}\cos\\ \sin\end{matrix}\, n\theta$, we have a means of evaluating the Fourier coefficients by measuring the angle turned through by the cylinder, provided we have a mechanism for imparting the necessary displacement $\phi(\theta)d\theta$ to a point in the circumference of the disc and at the same time causing the sphere to move so that the distance of its point of contact from the centre of the disc is $f(\theta)$. The following is one mode of producing the requisite motions :—

The graph of $y=f(\theta)$ is traced on a sheet of paper which is wound on a cylinder C_1, and the graph of $y=\int_0^\theta \begin{matrix}\cos\\ \sin\end{matrix}\, n\theta d\theta$ is drawn on another sheet which is wound on an equal cylinder C_2. The axes of the cylinders are parallel to that of the integrating cylinder, and the y-axis on C_1 and C_2 is a generator in each case. Arrangements are made so that the two cylinders C_1 and C_2 rotate with equal angular velocities. A rod l_1 parallel to a generator of the cylinder C_1, is furnished with a tracer which is made to trace the curve on C_1, and has at the other end a fork arrangement which moves the sphere with it, the sphere being at the centre of the disc when the tracer is on the θ-axis. A rod l_2, parallel to a generator of C_2, is also furnished with a tracer which traces the curve on C_2, while the rod is geared directly to the circumference of the disc.

It is at once obvious that such a mechanism will give the desired motions to the disc and sphere. If, then, the cylinders C_1 and C_2 be rotated through an angle θ and the tracers be made to follow their respective curves, the integrating cylinder records a reading proportional to $\int_0^\theta f(\theta) \begin{matrix}\cos\\ \sin\end{matrix}\, n\theta d\theta$, and thus if the cylinders C_1 and C_2 make a complete revolution, *i.e.* if θ goes from 0 to 2π, the reading of the integrating cylinder, if properly calibrated, will give $\frac{1}{\pi}\int_0^{2\pi} f(\theta) \begin{matrix}\cos\\ \sin\end{matrix}\, n\theta d\theta$, *i.e.* the value of the Fourier coefficient a_n or b_n.

It follows that such a mechanism as is described above is required for the determination of each coefficient in the Fourier expansion. In reality, instead of having one cylinder of the type C_1 for each coefficient, one cylinder of this type serves for the whole apparatus, the rod l_1 being attached by a fork arrangement to all the spheres of the mechanism. Further, the cylinders of type C_2 can be replaced by the following arrangement, which does not require that the curves $y=\int \begin{matrix}\cos\\ \sin\end{matrix}\, n\theta d\theta$ should be constructed. A crank communicates a simple harmonic angular motion of the proper period to the discs, while at the same time the cylinder C_1 moves with uniform speed.

In the final form, then, the instrument can be worked by one operator, who has to make the tracer of the arm l_1 follow the curve to be analysed on the cylinder C_1, and the readings of the various integrating cylinders give the coefficients.

Kelvin's Harmonic Analyser, which calculates by one operation the coefficients up to the third harmonic, has done useful work in the Meteoro-

logical Office in London. Owing to its complicated mechanism and weight it is practically a fixture in the room where it is used. Another instrument, which is also heavy and not conveniently portable, is that of Sommerfeld and Wiechert.[1] This instrument is not so complicated as that of Kelvin, but instead of calculating several coefficients, it calculates the coefficients successively, an operation being required for each. It possesses the advantage, however, that any number of the coefficients may be calculated, and also it avoids the simple harmonic motion and the consequent friction which such a mechanism entails. In this machine there are two essentially different processes performed, which in practice are carried out simultaneously. These are the construction of the curves $y=f(\theta)\,\begin{matrix}\cos\\ \sin\end{matrix}\,n\theta$ from the given curve $y=f(\theta)$, and, secondly, the integration of these new curves.

The Harmonic Analysers of Henrici

There are two instruments due to Professor Henrici [2] of London. The first of these was suggested to him by the Graphical Method of Clifford,[3] on which principle the construction of an instrument which evaluates the coefficients successively is based. A plane is required to perform a simple harmonic motion, and, as we have already pointed out, this is an objection in view of the mechanism that is required. To obviate the necessity for the simple harmonic motion, Professor Henrici devised another instrument based on the following theory:—

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_0^{2\pi} y\cos n\theta d\theta \\
&= \left[\frac{1}{n\pi}y\sin n\theta\right]_0^{2\pi} - \frac{1}{n\pi}\int_0^{2\pi}\sin n\theta dy \\
b_n &= \frac{1}{\pi}\int_0^{2\pi} y\sin n\theta d\theta \\
&= \left[-\frac{1}{n\pi}y\cos n\theta\right]_0^{2\pi} + \frac{1}{n\pi}\int_0^{2\pi}\cos n\theta dy.
\end{aligned}$$

If the curve to be analysed is continuous, the terms in the square brackets disappear. In this case the initial and final values of y are the same, *i.e.* referring to fig. 2, AA′=BB′ and the coefficients are given by

$$a_n = -\frac{1}{n\pi}\int_{\theta=0}^{\theta=2\pi}\sin n\theta dy$$

$$b_n = \frac{1}{n\pi}\int_{\theta=0}^{\theta=2\pi}\cos n\theta dy.$$

[1] *Dyck's Catalogue*, p. 214, 1892.
[2] *Phil. Mag.*, xxxviii. p. 110, 1894.
[3] *Proc. Lond. Math. Soc.*, v. 11, 1873.

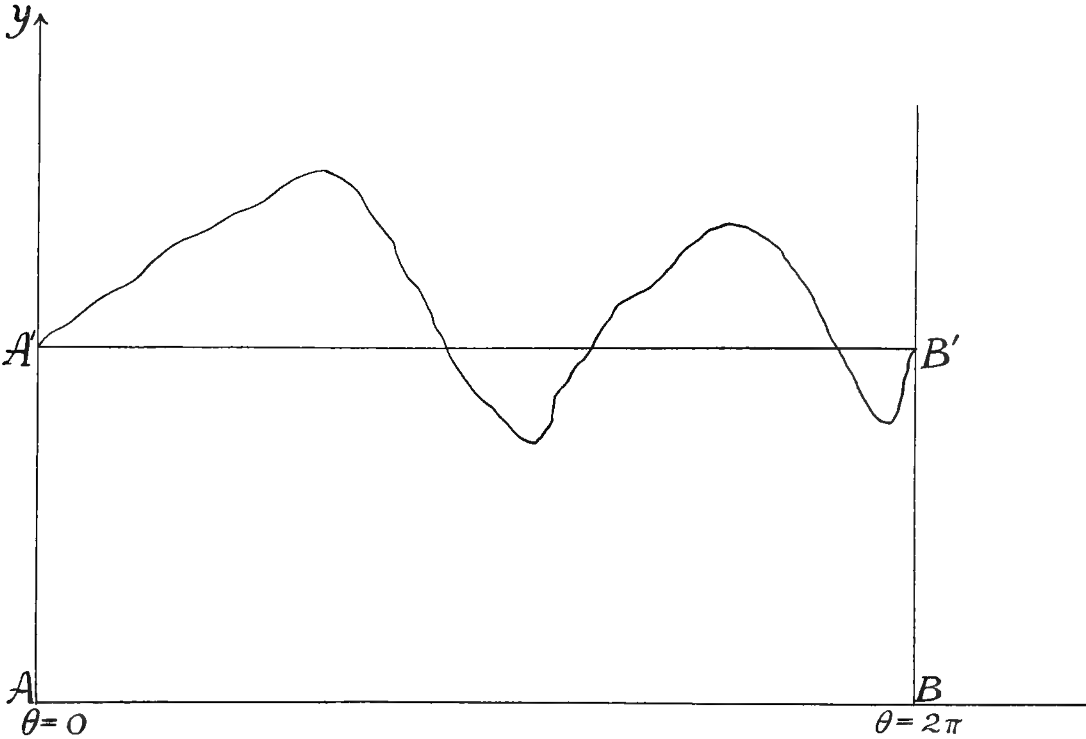

FIG. 2.

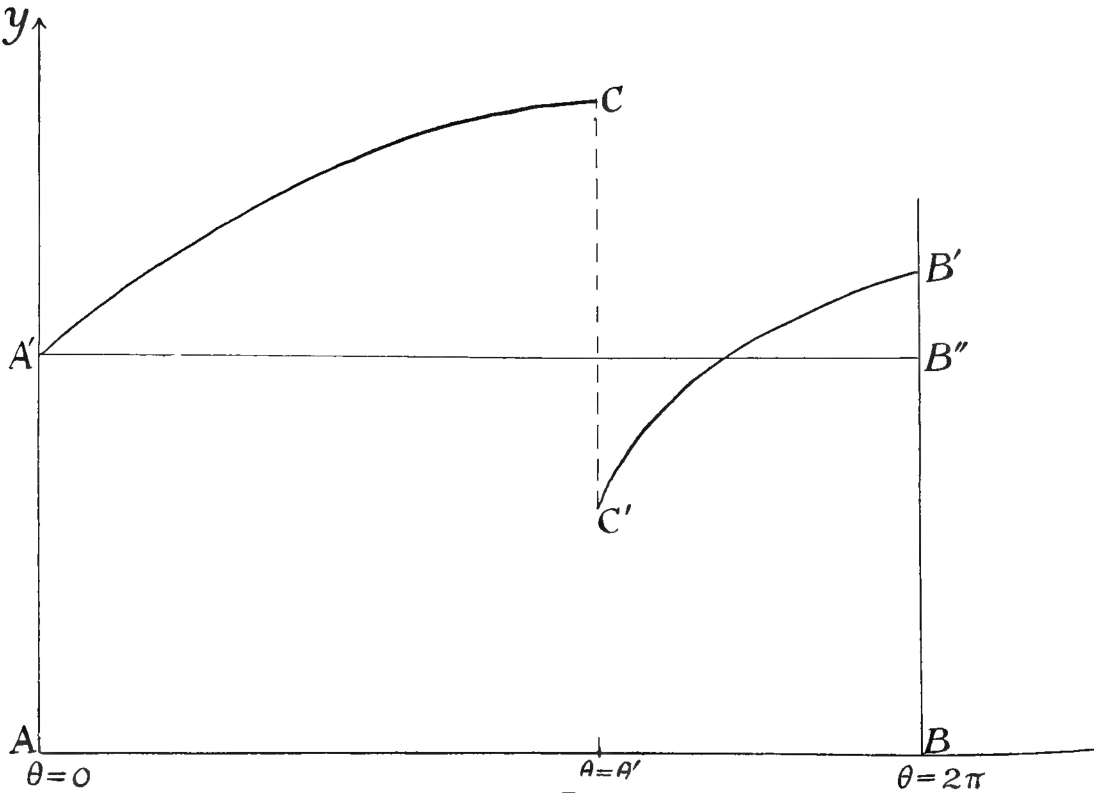

FIG. 3.

If the initial and final values of y be not the same (see fig. 3), and if in addition there be a discontinuity at the point C for which $\theta=\theta'$,

$$\pi a_n = \int_0^{\theta'} y \cos n\theta d\theta + \int_{\theta'}^{2\pi} y \cos n\theta d\theta.$$

integrating by parts we get

$$n\pi a_n = (y_1 - y_2) \sin n\theta' - \int_{\theta=0}^{2\pi} \sin n\theta dy,$$

where $y_1 - y_2 = C'C$.

So that in this case $-\frac{1}{n\pi}\int_{\theta=0}^{\theta=2\pi} \sin n\theta dy$ does not measure the coefficient a_n. If, however, the curve be made continuous by adding the portions CC′, B′B″ to the curve, as in the figure, we can easily prove that $-\frac{1}{n\pi}\int \sin n\theta dy$, where the integral is now taken along the continuous curve A′CC′B′B″, does measure the coefficient a_n. Hence, if the curve be discontinuous, we make it continuous as indicated, and the value of a_n is got by an integral of the same type as in the case of a continuous curve. Further, if the base line B′A′ in fig. 2 or B″A′ in fig. 3 be added to the path of integration, nothing is added to $\int \sin n\theta dy$ for $dy=0$, and now the integral is taken round a closed curve. Similar reasoning shows that $b_n = \frac{1}{n\pi}\int \cos n\theta dy$ where the integral is taken round the closed curve indicated. To effect the evaluation of these integrals —the Henrici integrals,—which are obviously of a different form from those of Cauchy, a tracer follows the curve, and at each instant the dy has to be multiplied by the $\begin{smallmatrix}\cos\\ \sin\end{smallmatrix} n\theta$, and the summation made of these contributions. This can be carried out by having two integrating wheels with their axes at right angles and making angles $n\theta$ and $\frac{\pi}{2}-n\theta$ with the y-axis and having the point of intersection of the axes capable of moving in a direction parallel to the y-axis. In the 1889 model of this instrument the curve $y=f(\theta)$ was traced on a cylinder mounted on a horizontal axis, the y-axis being along a generator. To a carriage which could move along a rail parallel to the axis of the generator was attached a vertical spindle forming part of a mechanism which enabled the axes of the integrating wheels to be turned through an angle $n\theta$ when the cylinder rotated through an angle θ. To the end of the spindle a tracer was attached which was constrained to follow the curve $y=f(\theta)$ by moving along the top generator of the cylinder, and hence the integrating wheels recorded the values of the coefficients a_n and b_n.

In the hands of Coradi the instrument has been greatly improved, and in its final form has attained a high degree of perfection, and, moreover, since a number of discs of different diameters and spindles have been inserted, it is

possible to get several pairs of coefficients by going over the curve once. Further details as to design can be got in Henrici's paper (*loc. cit.*) and *Dyck's Catalogue*.

Sharp's[1] *Harmonic Analyser*

A Fourier series

$$y = a_0 + a_1 \cos\theta + a_2 \cos 2\theta + \ldots \\ + b_1 \sin\theta + b_2 \sin 2\theta + \ldots$$

can be put in the form

$$y = c_0 + c_1 \sin(\theta + \alpha_1) + c_2 \sin(2\theta + \alpha_2) + \ldots + c_n \sin(n\theta + \alpha_n) + \ldots$$

where

$$c_0 = a_0,\ c_n \sin\alpha_n = a_n \quad \text{and} \quad c_n \cos\alpha_n = b_n.$$

Sharp's instrument is designed to calculate the amplitude and phase c_n and α_n of the different harmonics, and does so by evaluating the Henrici integrals.

If we have a wheel mounted on an axis which is constrained to move so as always to be parallel to the base of the curve to be analysed, and a tracer attached to a point of the axis, then the distance rolled over by the wheel is equal to the displacement of the tracer in the direction of the y-axis, *i.e.* is equal to dy.

This wheel rests on a circular disc which forms part of a mechanism consisting of three discs, and two wheels on an axle, the discs being coupled by means of keys and slots, and one of them being driven by the axle by means of bevel wheels, the discs, as Sharp points out, being kinematically equivalent to Oldham's coupling for the transmission of motion between two parallel shafts. In an actual instrument wheels and rails are substituted for the keys and slots, the object being to minimise frictional resistance. On the disc, on which rests the wheel to whose axis the tracer is attached, a secondary curve is traced by the point of contact of the wheel. By means of the mechanism this curve is such that an element of its length is equal to dy while it makes an angle θ with the y-axis. If P, P′ be two consecutive points on the curve to be analysed—the primary curve—and p, p' the corresponding points on the secondary curve, $p\,p' = dy$, and if p_1 be the projection of p' on a line through p parallel to the y-axis

$$p\,p_1 = \cos\theta dy \\ p_1 p' = \sin\theta dy.$$

Thus if a and b be initial and final position of the point p, we see that the projection of the curve apb on lines perpendicular and parallel to the y-axis respectively give $\int \sin\theta dy$ and $\int \cos\theta dy$. Hence if P makes a circuit of the primary curve, we have a means of finding the values of the coefficients a_1 and b_1, *i.e.* we have got the part $a_1 \cos\theta + b_1 \sin\theta$ of the Fourier expansion. This part can be put in the form $c_1 \sin(\theta + \alpha_1)$ where c_1 and α_1 are respectively the amplitude and phase of the first harmonic. The c_1 and α_1 could be measured just as readily by this method as a_1 and b_1. For the first harmonic the gearing is such that the disc makes a complete revolution, while the

[1] *Phil. Mag.*, xxxviii. 121, 1894.

tracer describes a complete period of the primary curve. By arranging that wheels of correspondingly smaller diameter can be substituted for the wheels of the wheel and axle arrangement above described, the amplitudes and phases of the second, third . . . harmonics can be obtained in succession.

Yule's[1] *Harmonic Analyser*

XX_1 is a line parallel to the base OB of the curve OAB to be analysed, and is capable of motion in the direction of the y-axis only. A circular disc whose centre is P is constrained to roll on the line XX_1 and have its centre

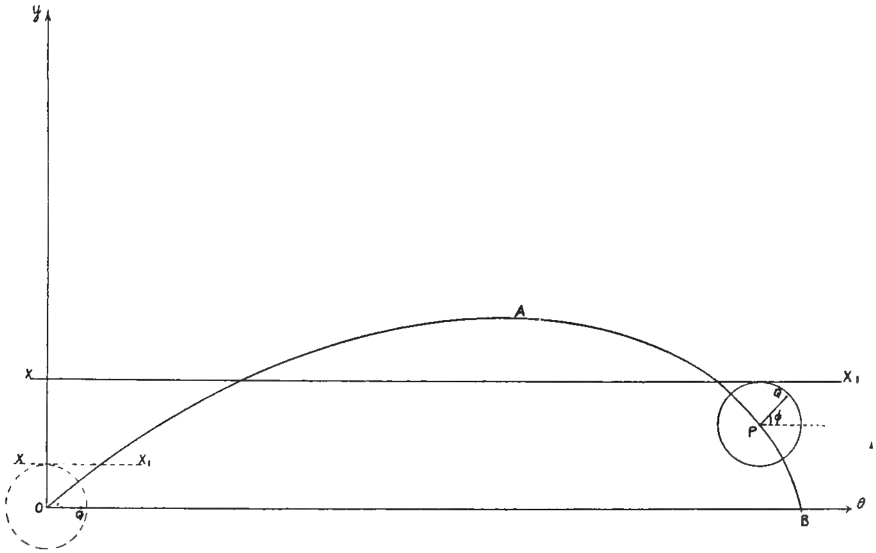

FIG. 4.

always on the curve. When P is at O, a point Q (not necessarily inside the disc) is marked on the diameter which coincides with OB and to the right of O. If the circumference of the disc $=\dfrac{2\pi}{n}$ where n is an integer, then if (θ, y) and (Θ, Y) be the co-ordinates of P and Q respectively, we have

$$\Theta=\theta+r\cos\phi,\ Y=y+r\sin\phi,$$

where r is the length PQ and ϕ is the angle turned through by PQ.

Now, since the radius of the disc is $\dfrac{1}{n}$, the angle ϕ is given by $\dfrac{1}{n}\phi=\theta$, for θ is the distance along the θ-axis that the point of contact of the disc has travelled, *i.e.*

$$\phi=n\theta.$$

Hence

$$\Theta=\theta+r\cos n\theta,$$
$$Y=y+r\sin n\theta.$$

[1] *Phil. Mag.*, xxxix. 367, 1895; *The Electrician*, 22nd March 1895.

The area of the curve traced by Q is given by

$$\int Y d\Theta = \int (y + r \sin n\theta) d(\theta + r \cos n\theta)$$
$$= \int y d\theta + r \int y d(\cos n\theta)$$
$$+ r \int \sin n\theta d\theta + r^2 \int \sin n\theta d(\cos n\theta).$$

If P be made to describe the curve and come back again to O along the θ-axis, we see that the last two integrals vanish, while nothing is added to the first two by the path along the θ-axis for $y=\mathrm{O}$, and hence the area of the closed curve traced by $\mathrm{Q} = \mathrm{S} - rn\int_0^{2\pi} y \sin n\theta d\theta$, where S is the area between the curve OAB and the θ-axis.

In the same way, if, when P is at O, we mark a point Q on the diameter perpendicular to the base line and above O, we get the area now traced by Q as $\mathrm{S} - rn\int_0^{2\pi} y \cos n\theta d\theta$. It follows from this, that, knowing S and the areas of the curves traced by Q, we can determine the coefficients a_n and b_n. These areas can be measured by a planimeter, and the areas traced by Q are measured at once by having the tracing point of a planimeter attached to Q, while P follows the curve in the manner indicated. This is the principle on which Yule's instrument is based. The actual apparatus consists of a rolling parallel ruler with a rack cut along one edge, and a number of toothed wheels which correspond to the disc indicated in the theory. He has had constructed four discs having respectively 240, 120, 80, and 60 teeth, and thus four harmonics can be obtained, the base line being 30 centimetres and the rack being in consequence cut 8 teeth to the centimetre. In the disc with 240 teeth there are cut three windows: the centre window has a black dot, the tracing point P, while the two other windows have fiducial marks that form with the point P a base line which allows the disc to be set in any desired position. On a radius perpendicular to this base line is a conical hole Q which receives the tracing point of the planimeter used for the evaluation of the area traced by Q. From the theory it will be seen that PQ must be the same length for all the discs, and in the actual instruments is $10/\pi$ centimetres.

Hence the small discs must be provided with an arm, on the top of which is a hole to receive the tracer of the planimeter, and this arm must be clear of the rack when the wheel is in gear, the windows being arranged as in the first disc. The planimeter used must have a tracer which is adjustable vertically. The coefficients are determined each by a separate operation, and the curve to be analysed must be drawn to such a scale that the base line is the standard length.

Michelson and Stratton's Harmonic Analyser

Michelson and Stratton[1] have described a form of analyser which depends on the use of springs. The essential parts are arranged as follows:—S is a large spring, and *s* is one of a number *n* of small springs, which are attached

[1] *Phil. Mag.*, xlv. 85, 1898.

respectively to the opposite ends B and A of a lever whose fulcrum is O. This lever is a prolongation of the horizontal diameter of a cylinder which is capable of rotating about its axis. The small springs are attached to a bar at right angles to the plane of the paper at equal distances apart. An eccentric at P_1 produces a harmonic motion which is communicated to the end C_1 of the small spring s by a lever F_1H_1 having a fulcrum at G_1 and jointed at F_1 and H_1, a rod R_1 and a lever C_1D_1 jointed to R_1, and having a fulcrum at D_1. By means of this mechanism a motion can be communicated from P_1 to the end of the spring s; this mechanism is repeated for each of the n small

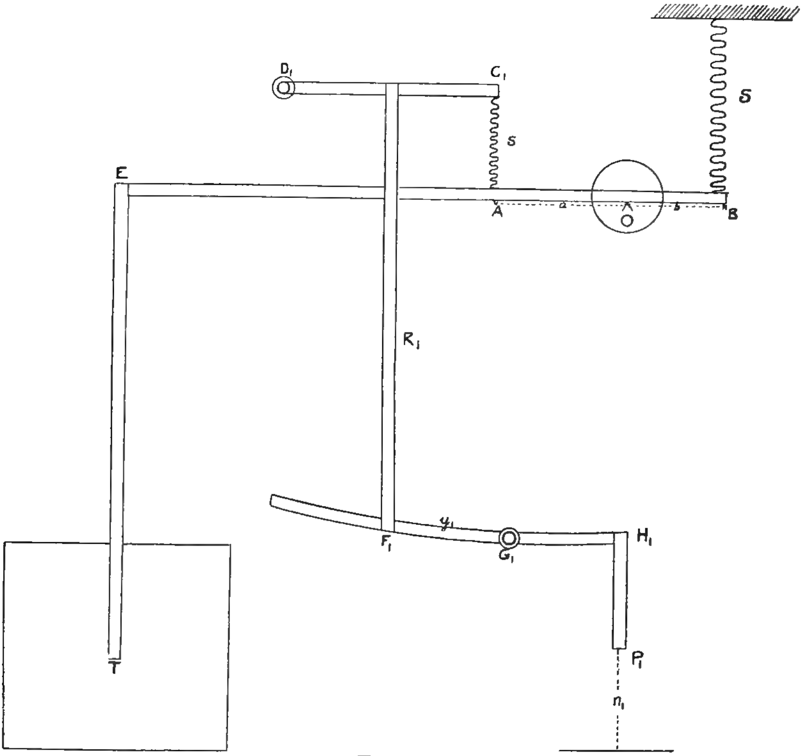

FIG. 5.

springs. E communicates the resultant motion by means of a style ET connected to it, the style registering its displacement on a slide which moves with a speed proportional to the angular speed of a cone formed of a number of gear wheels on its axis, one of the wheels being geared to each eccentric. The wheels have a number of teeth such that when the first eccentric makes one revolution the others make 2, 3, . . . n revolutions. If this cone be turned, C_1, C_2, C_3 have motions corresponding to $\cos\theta$, $\cos 2\theta$, $\cos 3\theta$, . . . and amplitudes depending on the distances $y_1, y_2, y_3,$ . . . where y_1 is the distance between the points F_1 and G_1, etc. To obtain motions corresponding to $\sin\theta$, $\sin 2\theta$, $\sin 3\theta$, . . . the eccentrics, disconnected from the gear wheels of the cone, are turned through 90° and again brought into gear.

If $l+x$ be the stretched length of the spring s

$L+y$,, ,, ,, S

and a, b the respective distances of s, S from the axis of the cylinder, it can be shown that

$$y=\frac{\Sigma x}{n\left(\frac{l}{\mathrm{L}}+\frac{a}{b}\right)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

it being assumed that Hooke's law holds. Hence it follows that the resultant motion is proportional to the algebraic sum of the motions of the small springs.

If P_r moves through a distance η_r, then $x_r=\lambda\eta_r y_r$ where λ is a constant, and thus if all the points P corresponding to all the springs s be made to lie on a curve $\eta=f(\theta)$, then all the C's lie on a curve $x=\lambda f(\theta)$ if y_1, y_2, y_3 . . . be each unity.

Now, if d be the distance between two consecutive springs s, the area of the curve on which the C's lie is approximately

$$\Sigma xd=d\Sigma x=\mu y \text{ by } (1),$$

where μ is a constant, and thus the y measures the area of the curve on which the C's lie.

If the P's be made to move by means of the eccentrics already described, and the y's, whose lengths can be varied, be proportional to the amplitudes a_1, a_2, . . . then the point T draws a curve whose equation is $\mathrm{Y}=a_1\cos\theta+a_2\cos 2\theta+$ . .

To use the instrument as an harmonic analyser, the y's must be so adjusted as to be proportional to the ordinates of the curve to be analysed, the P's having the same motion as before. It can be seen that the tracer now describes a curve from which the Fourier coefficients can be got by measurement of ordinates at equal distances.

For a full account of this instrument the reader is referred to Henrici's article, "Calculating Machines," p. 981, *Encyclopædia Britannica*, 11th edition, where it is also shown that the instrument can be used as an integraph.

Harmonic Analyser of Boucherot[1]

Let P be any point of the curve to be analysed, and suppose its co-ordinates are (θ, y). Through P draw a line PQ of length l, making an angle $n\theta$ with the θ-axis, and construct an isosceles triangle PQR with its base PR parallel to the θ-axis. The co-ordinates of R are $(\theta+2l\cos n\theta, y)$. As P traces the curve to be analysed, R traces another curve whose area is

$$\int_0^{2\pi} yd(\theta+2l\cos n\theta)=\int_0^{2\pi} yd\theta-2ln\int_0^{2\pi} y\sin n\theta d\theta.$$

Thus if $\int_0^{2\pi} yd\theta=0$, *i.e.* if the mean ordinate of the original curve be zero, we see that the point R traces a curve whose area gives us the value of the coefficient b_n. The area is got by having the tracing point of a planimeter at the point R, while the point P describes a complete circuit of the original

[1] Morin, *Les Appareils d'Intégration*, pp. 179–183, 1913.

curve. As the planimeter does not distinguish between positive and negative areas, the base line must be so arranged that the ordinate y is always positive.

The corresponding area requires to be determined when the line PQ is inclined at an angle $n\theta+\frac{\pi}{2}$ to the θ-axis instead of $n\theta$. Knowing the values of these two areas, we can calculate a_n and b_n. If the mean ordinate of the curve is not zero, we can determine $\int_0^{2\pi} y d\theta$, the area of the original curve, by means of a planimeter.

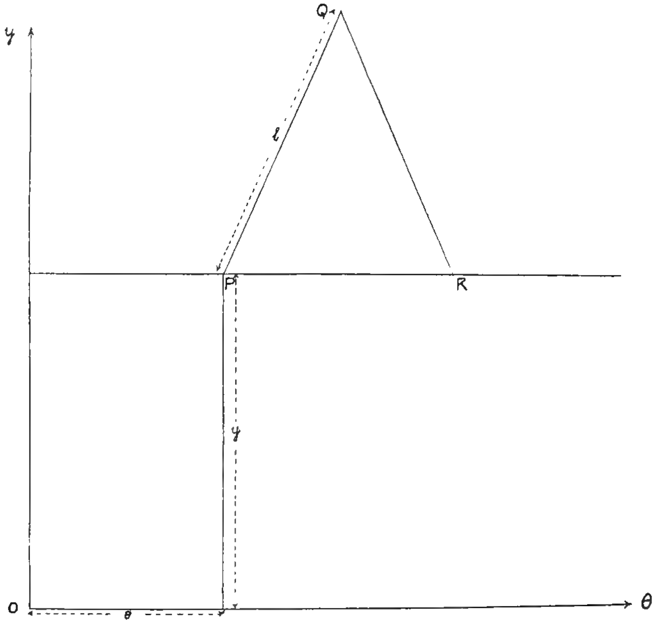

FIG. 6.

The essential parts of the apparatus consist of two rods at right angles to each other. One of these is fixed and forms the y-axis, while one end of the other is capable of moving along the first. PR is part of this latter rod, and at P, the tracing point, there is an arrangement by which, when P moves through a distance θ along the rod, the arm PQ turns through an angle $n\theta$, where n may have the values 1, 2, 3, . . . successively.

Mader's Harmonic Analyser[1]

If the Fourier series be given in the form

$$f(x) = a_0 + a_1 \cos \frac{\pi x}{a} + a_2 \cos 2\frac{\pi x}{a} + a_3 \cos 3\frac{\pi x}{a} + \quad . \quad . \quad + a_n \cos n\frac{\pi x}{a} + \quad . \quad . \quad .$$
$$+ b_1 \sin \frac{\pi x}{a} + b_2 \sin 2\frac{\pi x}{a} + \quad . \quad . \quad + b_n \sin n\frac{\pi x}{a} + \quad . \quad .$$

[1] *Elektrotech. Zeit.*, xxxvi., 1909. For the theory see A. Schreiber, *Phys. Zeit.*, xi. 354, 1910.

then

$$a_0 = \frac{1}{2a}\int_0^{2a} y\,dx,$$

$$a_n = \frac{1}{a}\int_0^{2a} y \cos\left(n\frac{\pi x}{a}\right)dx,$$

$$b_n = \frac{1}{a}\int_0^{2a} y \sin\left(n\frac{\pi x}{a}\right)dx.$$

Like Henrici's first instrument, the construction of this instrument is based on Clifford's graphical method. The instrument consists of two carriages, an upper and a lower; the latter of these is only capable of motion in a straight line, which is here taken as the y-axis, and carries an angle lever

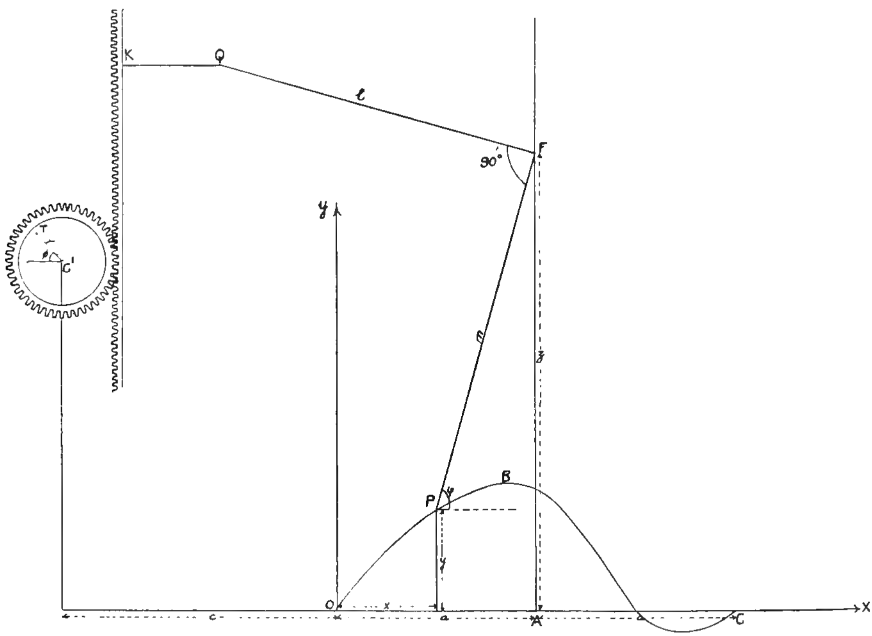

FIG. 7.

PFQ, consisting of two arms at right angles to each other. F is fixed to the lower carriage, and thus moves only in the direction of the y-axis. At P is attached a tracer which is made to follow the curve to be analysed, and the distance of P from F can be varied. Q can move along a line QK which is parallel to the x-axis, and QK is part of the upper carriage which runs on the lower carriage, and when the angle lever turns about F the upper carriage moves relatively to the lower one. A toothed edge attached to the upper carriage engages a toothed disc attached to the lower carriage, so that the rotation of this disc measures the relative displacement of the upper and lower carriages. The tracing point of an ordinary planimeter is fitted into one or other of two depressions in this toothed wheel, these depressions being at equal distances from the centre of the disc and subtending a right angle at it. The reading of the planimeter, which is got when the operations, to be pre-

sently described, have been carried out, gives a_n or b_n, according as the tracing point of the planimeter has been fitted into one or other of the depressions on the disc. By substituting different discs the coefficients of the different harmonics can be obtained.

The curve OBC to be analysed is placed so that the middle point A of its base line OC is such that AF is parallel to the y-axis, and the length of the arm PF is so adjusted that when the tracing point P is at O the depression T in which the tracing point of the planimeter is fitted lies on a diameter of the disc which is parallel to the x-axis and coincides with a mark on the toothed edge.

If the co-ordinates of P be (x, y),

$$\left.\begin{aligned} x &= a - m\cos\psi \\ y &= z - m\sin\psi \end{aligned}\right\} \qquad (1)$$

where a is the length OA, z is the length FA, m is the length of the arm FP, and ψ is the angle FP makes with the x-axis.

If (ξ, η) be the co-ordinates of T and $(-c, \eta_0)$ the initial co-ordinates of C', the centre of the disc, and z_0 the initial value of z,

$$\left.\begin{aligned} \xi &= -(c + r\cos\phi) \\ \eta &= \eta_0 + r\sin\phi + z - z_0 \end{aligned}\right\} \qquad (2)$$

where r is the length C'T and ϕ is the angle turned through by the disc.

If l be the length of the arm FQ,

$$l(\cos\psi - \cos\psi_0) = \mathrm{R}\phi \qquad (3)$$

where ψ_0 is the initial value of ψ and R is the radius of the disc.

$$\therefore \qquad \mathrm{R}\phi = -\frac{lx}{m},$$

for from (1)

$$x - x_0 = -m(\cos\psi - \cos\psi_0),$$
$$i.e.\ x = -m(\cos\psi - \cos\psi_0) \text{ since } x_0 = 0.$$

The area traced out by T is $\int(\eta - \eta_0)d\xi = \int(r\sin\phi + z - z_0)d\xi$, where the integral is taken round the closed curve traced by T as P describes the curve OBC and returns to O along the base line,

$$= \int(r\sin\phi + z)d\xi, \text{ since } \int z_0 d\xi = 0 \text{ when taken round a closed curve,}$$

$$= \int(r\sin\phi + z)r\sin\phi d\phi \text{ using (2)}$$

$$= \int\left\{r\sin\left(-\frac{lx}{\mathrm{R}m}\right) + y + m\sin\psi\right\} r\sin\left(-\frac{lx}{\mathrm{R}m}\right)\left(-\frac{l}{\mathrm{R}m}\right)dx,$$

$$= \frac{rl}{\mathrm{R}m}\int y\sin\frac{lx}{\mathrm{R}m}dx - \frac{rl}{\mathrm{R}m}\int\left(r\sin\frac{lx}{\mathrm{R}m} - m\sin\psi\right)\sin\frac{lx}{\mathrm{R}m}dx.$$

Now, since $\sin\psi$ can be expressed as a function of x only, the second of the

integrals vanishes when taken round a closed curve. Hence the area traced by T is

$$\frac{rl}{\mathrm{R}m}\int y \sin \frac{lx}{\mathrm{R}m} dx, \text{ taken round the closed curve traced by P,}$$

$$= \frac{rl}{\mathrm{R}m}\int_0^{2a} y \sin \frac{lx}{\mathrm{R}m} dx, \text{ since } y = 0 \text{ along the } x\text{-axis.}$$

If the radius of the disc be such that $\frac{l}{\mathrm{R}m} = \frac{n\pi}{a}$, then the planimeter records the value of b_n. If the tracing point of the planimeter is placed on the depression which is initially on a diameter perpendicular to the x-axis, the value of a_n is got. Discs of different diameters are provided with the instrument which enables the coefficients a_1, b_1; a_2, b_2; . . . to be determined, a_0 being measured directly by means of the planimeter.

II. Arithmetical Methods

In the Fourier series

$$\begin{aligned} y = a_0 + a_1 \cos \theta + a_2 \cos 2\theta + \quad & \quad + a_n \cos n\theta + \\ + b_1 \sin \theta + b_2 \sin 2\theta + \; . \quad & \quad + b_n \sin n\theta + \end{aligned}$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} y\,d\theta, \quad a_n = \frac{1}{\pi}\int_0^{2\pi} y \cos n\theta\,d\theta, \quad b_n = \frac{1}{\pi}\int_0^{2\pi} y \sin n\theta\,d\theta;$$

hence we see that a_0 is the mean ordinate, while a_n is twice the mean value of the product of the ordinate corresponding to θ and $\cos n\theta$, and similarly b_n is twice the mean value of the product of the ordinate corresponding to θ and $\sin n\theta$.

An arithmetical mode, therefore, of finding the coefficients approximately consists in multiplying a finite number of ordinates by the cosine of the corresponding angle, or by the cosine of twice the corresponding angle, and so on. If the mean of these products be taken over a whole period, we obtain the values of $2a_1$, $2a_2$, . . . and multiplying by sines instead of cosines we obtain the values of $2b_1$, $2b_2$. . . .

In practical applications we are given a limited number of ordinates of a curve, and the problem is to determine its equation in the form

$$\begin{aligned} y = a_0 + a_1 \cos \theta + \; . \; . \; . \\ + b_1 \sin \theta + \quad . \; . \end{aligned}$$

In practice a selected number of equidistant ordinates are taken throughout the period, and we shall deal with the case of twenty-four equidistant ordinates for the sake of simplicity and concreteness, and also in view of the fact that a twenty-four ordinate method is of importance in numerous cases that occur in meteorological, astronomical, and other investigations, in which a curve has to be analysed into its harmonic components, though it will be seen that the reasoning employed is perfectly general. The period 0 to 2π is divided into twenty-four equal parts by taking the points $\theta = 0$, $\theta = \frac{\pi}{12}$,

$\theta=2\frac{\pi}{12}$ . . . $\theta=23\frac{\pi}{12}$, the ordinates at these points being denoted by u_0, u_1, u_2 . . . u_{23}. We may take as an approximate value for u

$$u=a_0+a_1\cos\theta+a_2\cos 2\theta+ \quad . \; +a_{12}\cos 12\theta$$
$$+b_1\sin\theta+b_2\sin 2\theta+ \quad +b_{11}\cos 11\theta,$$

or shortly

$$u=a_0+\sum_{p=1}^{p=12}a_p\cos p\theta+\sum_{p=1}^{p=11}b_p\sin p\theta.$$

To determine the twenty-four constants a_0, a_1, . . . b_{12}, b_1 . b_{11}, we have the following twenty-four equations :

$$u_0=a_0+a_1+a_2+ \quad . \; +a_{12} \qquad (1)$$

$$u_1=a_0+\sum_{p=1}^{p=12}a_p\cos\frac{p\pi}{12}+\sum_{p=1}^{p=11}b_p\sin\frac{p\pi}{12}. \qquad (2)$$

$$u_2=a_0+\sum_{p=1}^{p=12}a_p\cos\frac{2p\pi}{12}+\sum_{p=1}^{p=11}b_p\sin\frac{2p\pi}{12} \qquad (3)$$

.

.

.

$$u_{23}=a_0+\sum_{p=1}^{p=12}a_p\cos\frac{23p\pi}{12}+\sum_{p=1}^{p=11}b_p\sin\frac{23p\pi}{12}. \qquad . \; (24)$$

These equations may be solved by various methods, but the following method [1] is convenient. To determine the coefficient a_r, say, multiply the equations (1) . . . (24) in order by

$$1, \cos\frac{r\pi}{12}, \cos\frac{2r\pi}{12} \quad . \; \cos\frac{23r\pi}{12}$$

respectively ; adding these equations we get

$$\sum_{p=0}^{p=23}u_p\cos\frac{pr\pi}{12}=a_r\sum_{p=0}^{p=23}\cos^2\frac{pr\pi}{12},$$

since the sums of the trigonometrical series by which the other coefficients are multiplied are each zero.

Hence, since

$$\sum_{p=0}^{p=23}\cos^2\frac{pr\pi}{12}=12\;(r=1,\,2, \quad 11)$$
$$\text{and}=24\;(r=0,\,12)$$
$$\therefore \quad a_r=\frac{1}{12}\sum_{p=0}^{p=23}u_p\cos\frac{pr\pi}{12}(r=1, \quad 11)$$
$$\text{and}=\frac{1}{24}\sum_{p=0}^{p=23}u_p\cos\frac{pr\pi}{12}(r=0,\,12).$$

[1] See Gibson's *Introduction to the Calculus*, p. 130, 1906.

In the same way, multiplying by sines, we get

$$b_r = \frac{1}{12}\sum_{p=0}^{p=23} u_p \sin\frac{pr\pi}{12} (r = 1, \quad 11).$$

It should be noticed that the value of a_{12} cannot be immediately deduced from Cauchy's integrals for the coefficients. In the case in which all the coefficients $a_0, a_1, a_2, \quad . a_{12}$; $b_1, b_2, . \quad . b_{11}$ are to be determined, then, as we have seen, we have as many equations as coefficients. It is interesting to note that, since in many cases only the first few coefficients are important, the method of Least Squares might be applied, as there are now more equations than unknowns. It is easy to show that the values of these coefficients as determined by this Least Square method are the same as those got for these coefficients by solving all the twenty-four equations used above for the determination of all the twenty-four coefficients.

The greater the number of ordinates used, the greater will be the accuracy of the results obtained; on the other hand, an increase in the number of ordinates taken involves a very considerable increase in the amount of arithmetical work to be performed. The arithmetical labour involved is diminished if we consider that as θ increases from 0 to 2π both the cosine and sine pass four times through the same numerical value, two of these values being positive and two negative, and thus certain of the ordinates, if these be taken at equal distances, require to be multiplied by the same quantity. Various schemes, forms or schedules have been drawn out in which the amount of labour in performing the operations necessary for obtaining the coefficients has been very much reduced.

Strachey [1] has drawn out tables and formulæ to facilitate the computation of harmonic coefficients, particularly in reference to meteorological data in which there are hourly readings taken throughout the day or daily readings taken throughout the year. In one of the methods described by him he obtains the most probable values of the several harmonic coefficients from the series of observed values by employing the method of Least Squares. Many other schemes have been devised, and a number of these are based on those of Runge.[2] His method involves the multiplication by cosines and sines of angles, but instead of dealing with single ordinates, the latter are collected where possible and the operation of multiplication carried out on groups of ordinates; hence Silvanus Thompson terms the device "grouping." Runge's method deals both with the even and odd harmonics, and he has propounded modes of dealing with twelve, twenty-four, and thirty-six ordinates.

Still confining ourselves to the case of twenty-four ordinates being given, his scheme is based on the following considerations. The equations which we have already obtained for the coefficients are :—

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[1] *Hourly Readings*, 1884 (Meteorological Council), pt. iv., pub. in 1887; *Proc. Roy. Soc.*, xlii. 61–79.

[2] *Zeit. f. Math. u. Phys.*, xlviii. 443–456, 1903; lii. 117–123, 1905; *Erläuterung des Rechnungsformulars, u.s.w.*, Braunschweig, 1913.

$$
\begin{aligned}
24a_0 &= u_0 + u_1 + u_2 + \quad \ldots \quad + u_{23} \\
24a_{12} &= u_0 - u_1 + u_2 - \quad \ldots \quad - u_{23} \\
12a_1 &= u_0 + u_1 \cos 15^\circ + u_2 \cos 30^\circ + \quad \ldots \quad + u_{23} \cos 345^\circ \\
12a_2 &= u_0 + u_1 \cos 30^\circ + u_2 \cos 60^\circ + \quad \ldots \quad + u_{23} \cos 690^\circ \\
&\ldots \\
12b_1 &= u_1 \sin 15^\circ + u_2 \sin 30^\circ + \quad \ldots \quad + u_{23} \sin 345^\circ \\
12b_2 &= u_1 \sin 30^\circ + u_2 \sin 60^\circ + \quad \ldots \quad + u_{23} \sin 690^\circ. \\
&\ldots
\end{aligned}
$$

Arrange the u's in two rows as follows :—

	u_0	u_1	u_2	. . .	u_{11}	u_{12}
		u_{23}	u_{22}	. . .	u_{13}	
Add the rows .	v_0	v_1	v_2	. . .	v_{11}	v_{12}
Subtract the rows .		w_1	w_2	. . .	w_{11}	

$$
\begin{aligned}
\therefore \quad 24a_0 &= v_0 + v_1 + \quad \ldots \quad + v_{12} \\
12a_1 &= v_0 + v_1 \cos 15^\circ + v_2 \cos 30^\circ + \quad \ldots \quad + v_{12} \cos 180 \\
&\text{since} \quad \cos 345^\circ = \cos 15^\circ \text{ etc.} \\
12a_2 &= v_0 + v_1 \cos 30^\circ + v_2 \cos 60^\circ + \quad \ldots \quad + v_{12} \cos 360^\circ. \\
&\ldots \\
12b_1 &= w_1 \sin 15^\circ + w_2 \sin 30^\circ + \quad \ldots \quad + w_{11} \sin 165^\circ \\
12b_2 &= w_1 \sin 30^\circ + w_2 \sin 60^\circ + \quad \ldots \quad + w_{11} \sin 330^\circ. \\
&\ldots
\end{aligned}
$$

Arrange the v's in two rows as follows :—

	v_0	v_1	. . .	v_5	v_6
	v_{12}	v_{11}	. . .	v_7	
Add the rows . .	p_0	p_1	. . .	p_5	p_6
Subtract the rows .	q_0	q_1	. . .	q_5	

$$
\begin{aligned}
\therefore \quad 24a_0 &= p_0 + p_1 + \quad \ldots \quad + p_6 \\
12a_1 &= q_0 + q_1 \cos 15^\circ + \quad \ldots \quad + q_5 \cos 75^\circ \\
12a_2 &= p_0 + p_1 \cos 30^\circ + \quad \ldots \quad + p_6 \cos 180^\circ. \\
&\ldots
\end{aligned}
$$

Arrange the w's in two rows :—

	w_1	w_2	. . .	w_5	w_6
	w_{11}	w_{10}	. . .	w_7	
Add the rows . .	r_1	r_2	. . .	r_5	r_6
Subtract the rows	s_1	s_2	. . .	s_5.	

$$
\begin{aligned}
\therefore \quad 12b_1 &= r_1 \sin 15^\circ + r_2 \sin 30^\circ + \quad \ldots \quad + r_6 \sin 90^\circ \\
12b_2 &= s_1 \sin 30^\circ + s_2 \sin 60^\circ + \quad \ldots \quad + s_5 \sin 150^\circ \\
&\ldots
\end{aligned}
$$

Arrange the p's in two rows :—

$$\begin{array}{lcccc} & p_0 & p_1 & p_2 & p_3 \\ & p_6 & p_5 & p_4 & \\ \hline \text{Add the rows} & l_0 & l_1 & l_2 & l_3 \\ \text{Subtract the rows} \quad . & m_0 & m_1 & m_2. & \end{array}$$

$$\begin{aligned} \therefore \quad 24a_0 &= l_0 + l_1 + l_2 + l_3 \\ 12a_2 &= m_0 + m_1 \cos 30^\circ + m_2 \cos 60^\circ \\ 12a_4 &= l_0 + l_1 \cos 60^\circ + l_2 \cos 120^\circ + l_3 \cos 180^\circ \\ 12a_6 &= m_0 + m_1 \cos 90^\circ + m_2 \cos 180^\circ \\ 12a_8 &= l_0 + l_1 \cos 120^\circ + l_2 \cos 240^\circ + l_3 \cos 360^\circ \\ 12a_{10} &= m_0 + m_1 \cos 150^\circ + m_2 \cos 300^\circ \\ 24a_{12} &= l_0 - l_1 + l_2 - l_3. \end{aligned}$$

Arrange the s's in two rows :—

$$\begin{array}{lccc} & s_1 & s_2 & s_3 \\ & s_5 & s_4 & \\ \hline \text{Add the rows} \quad . \quad . & k_1 & k_2 & k_3 \\ \text{Subtract the rows} \quad . & n_1 & n_2 & \end{array}$$

$$\begin{aligned} \therefore \quad 12b_2 &= k_1 \sin 30^\circ + k_2 \sin 60^\circ + k_3 \sin 90^\circ \\ 12b_4 &= n_1 \sin 60^\circ + n_2 \sin 120^\circ \\ 12b_6 &= k_1 \sin 90^\circ + k_2 \sin 180^\circ + k_3 \sin 270^\circ \\ 12b_8 &= n_1 \sin 120^\circ + n_2 \sin 240^\circ \\ 12b_{10} &= k_1 \sin 150^\circ + k_2 \sin 300^\circ + k_3 \sin 450^\circ. \end{aligned}$$

His twelve-ordinate scheme, which affords a sufficiently accurate result for many cases that occur, will be given here. For the necessarily more complex schemes for twenty-four and thirty-six ordinates, reference should be made to his memoirs (*loc. cit.*).

His schematic arrangement is as follows :—

$$\begin{array}{lccccccc} \text{Ordinates} \quad . \quad . & u_0 & u_1 & u_2 & u_3 & u_4 & u_5 & \\ & & u_{11} & u_{10} & u_9 & u_8 & u_7 & u_6 \\ \hline \text{Differences} \quad . \quad . & & w_1 & w_2 & w_3 & w_4 & w_5 & \\ \text{Sums} \quad . \quad . & v_0 & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array}$$

$$\begin{array}{lccc} & w_1 & w_2 & w_3 \\ & w_5 & w_4 & \\ \hline \text{Sums} \quad . \quad . & r_1 & r_2 & r_3 \\ \text{Differences} & s_1 & s_2 & \end{array} \qquad \begin{array}{lcccc} & v_0 & v_1 & v_2 & v_3 \\ & v_6 & v_5 & v_4 & \\ \hline \text{Sums} \quad . & p_0 & p_1 & p_2 & p_3 \\ \text{Differences} \quad . & q_0 & q_1 & q_2 & \end{array}$$

$$\begin{array}{lcc} & r_1 & q_0 \\ & r_3 & q_2 \\ \hline \text{Differences} \quad . & t_1 & t_2 \end{array} \qquad \begin{array}{lcc} & p_0 & p_1 \\ & p_2 & p_3 \\ \hline \text{Sums} \quad . & l_0 & l_1 \end{array}$$

	Sine Terms.			Cosine Terms.			
Multipliers .	1, 5	2, 4	3	1, 5	2, 4	3	0, 6
Sin 30°	r_1			q_2	$-p_2 \quad p_1$		
Sin 60° .	$\quad r_2$	$s_1 \quad s_2$		$\quad q_1$			
Sin 90° .	r_3		t_1	q_0	$p_0 \quad -p_3$	t_2	$l_0 \quad l_1$
Sum of first column	...	...		...	...		...
Sum of second column .	...	...		...	...		...
Sum	$6b_1$	$6b_2$	$6b_3$	$6a_1$	$6a_2$	$6a_3$	$12a_0$
Difference .	$6b_5$	$6b_4$		$6a_5$	$6a_4$		$12a_6$

The scheme is practically self-explanatory, but an example is given which will make the mode of procedure perfectly clear.

Ordinates . .	−18	−39	−39	−8	22	22	
		−1	15	12	14	10	11
Differences (w) .		−38	−54	−20	8	12	
Sums (v) .	−18	−40	−24	4	36	32	11

	−38	−54	−20
	12	8	
Sums (r) .	−26	−46	−20
Differences (s) .	−50	−62	

	−18	−40	−24	4
	11	32	36	
Sums (p) .	−7	−8	12	4
Differences (q) .	−29	−72	−60	

	−26	−29
	−20	−60
Differences (t)	−6	31

	−7	−8
	12	4
Sums (l)	5	−4

	Sine Terms.			Cosine Terms.			
Multipliers . .	1, 5	2, 4	3	1, 5	2, 4	3	0, 6
·5	−13			−30	−6 −4		
·866	−40	−43 −54		−62			
1	−20		−6	−29	−7 −4	31	5 −4
Sum of first column .	−33	−43		−59	−13		5
Sum of second column	−40	−54		−62	−8		−4
Sum . .	$-73=6b_1$	$-97=6b_2$	−6	$-121=6a_1$	$-21=6a_2$	31	$1=12a_0$
Difference	$7=6b_5$	$11=6b_4$	$=6b_3$	$3=6a_5$	$-5=6a_4$	$=6a_3$	$9=12a_6$

Result—

$$u=0·1-20\cos\theta-3·5\cos 2\theta+5·2\cos 3\theta-0·8\cos 4\theta+0·5\cos 5\theta+0·8\cos 6\theta$$
$$-12\sin\theta-16\sin 2\theta-\sin 3\theta+1·8\sin 4\theta+1·2\sin 5\theta.$$

A very concise and convenient twenty-four ordinate computing form, based on Runge's analysis, has been devised by Whittaker for use in his Mathematical Laboratory at the University of Edinburgh. The form is shown on the following pages. In the first column the twenty-four ordinates are written down, and other columns are provided for entering the ordinates when these have been increased or decreased by a constant quantity; this operation does not alter the values of any coefficient except a_0, which is increased or decreased by the same constant quantity. The advantage of this operation is that the manipulation of large numbers is avoided.

θ				
0	u_0			
15°	u_1			
30°	u_2			
45°	u_3			
60°	u_4			
75°	u_5			
90°	u_6			
105°	u_7			
120°	u_8			
135°	u_9			
150°	u_{10}			
165°	u_{11}			
180°	u_{12}			
195°	u_{13}			
210°	u_{14}			
225°	u_{15}			
240°	u_{16}			
255°	u_{17}			
270°	u_{18}			
285°	u_{19}			
300°	u_{20}			
315°	u_{21}			
330°	u_{22}			
345°	u_{23}			

(a)

(u_0 to u_{12}) ....

(u_{23} to u_{13}) ....

Sums (v_0 to v_{12}) ....

Diffs. (w_1 to w_{11}) ....

(b)

(v_0 to v_6) ....

(v_{12} to v_7) ....

(p_0 to p_6) ....

(q_0 to q_5) ....

(w_1 to w_6) ....

(w_{11} to w_7) ....

(r_1 to r_6) ....

(s_1 to s_5) ....

(c)

(s_1 to s_3) ....

(s_5, s_4) ....

Sums (k_1 to k_3) ....

Diffs. (n_1, n_2) ....

(p_0 to p_3) ....

(p_6 to p_4) ....

Sums (l_0 to l_3) ....

Diffs. (m_0 to m_2) ....

(d)

	n		k			m			l			
	1	2		 2			1					
			1		1	2		........-2	........-2	1	2	1
½ × previous line												
			....			....			....	....		
			3		........-3	0		0	0	........-3	0	3
·866 × first line	....	....		....			....					
Sum of 2nd column .	....		....			....			....		....	
·707 × previous line												
Sum of 1st column .	....		....			....			....		....	
Sum .	.... $=12b_4$		.... $=12b_2$		.... $=12b_6$	.... $=12a_2$		.... $=12a_6$	.... $=12a_4$		.... $=24a_0$	
Diff.	.... $=-12b_8$		.... $=-12b_{10}$			.... $=-12a_{10}$			.... $=-12a_8$		.... $=-24a_{12}$	

(e)

q						r					
........1						5					
........5						1					
........2	........1+5			........-2	........5+1	4	........5+1			........-4	........1+5
........4	........1−5	........-4	........1−5	4	........5−1	2	........5−1	2	........1−5	2	........1−5
....	....			....	....	....	....			....	....
........0	3	0	........-3	0	........-3	6	3	........-6	3	6	........-3
....	....			....	....	....	....			....	....
....		....		....		....		....		....	
....		....		....		....		....		....	
....		....		....		....		....		....	
.... $=12a_1$		.... $=12a_3$		.... $=12a_5$		.... $=12b_1$		.... $=12b_3$		.... $=12b_5$	
.... $=-12a_{11}$		.... $=-12a_9$		.... $=-12a_7$		.... $=12b_{11}$		.... $=12b_9$		.... $=12b_7$	

To enable the form to be reproduced conveniently here, it has been divided into the portions *a, b, c, d, e.* In the actual scheme *b* is immediately to the right of *a*, and *e* is to the right of *d*, and the instructions to the left of *d* also apply to the portion *e*.

The values of the coefficients are entered in the following table :—

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(0)
a													
b													
a^2													
b^2													
a^2+b^2													
$(a^2+b^2)^{\frac{1}{2}}$													
$1/b$													
a/b													
$\tan^{-1}a/b$													

Thus the result may be stated in either of the two forms :—

$$u = a_0 + a_1 \cos\theta + a_2 \cos 2\theta + \ldots + a_{12}\cos 12\theta + b_1 \sin\theta + b_2 \sin 2\theta + \ldots + b_{11}\sin 11\theta$$

or

$$u = c_0 + c_1 \sin(\theta + \alpha_1) + c_2 \sin(2\theta + \alpha_2) + \ldots + c_{11}\sin(11\theta + \alpha_{11}).$$

The following checks are applied :—

$$u_0 = a_0 + a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 + a_{10} + a_{11} + a_{12}.$$

$$\tfrac{1}{2}(u_1 - u_{23}) = \cdot 259\,(b_1 + b_{11}) + \tfrac{1}{2}(b_2 + b_{10}) + \cdot 707\,(b_3 + b_9) + \cdot 866\,(b_4 + b_8) + \cdot 966\,(b_5 + b_7) + b_6.$$

θ				
0°	u_0	149	−18	
15°	u_1	137	−30	
30°	u_2	128	−39	
45°	u_3	126	−41	
60°	u_4	128	−39	
75°	u_5	135	−32	
90°	u_6	159	−8	
105°	u_7	178	11	
120°	u_8	189	22	
135°	u_9	191	24	
150°	u_{10}	189	22	
165°	u_{11}	187	20	
180°	u_{12}	178	11	
195°	u_{13}	170	3	
210°	u_{14}	177	10	
225°	u_{15}	183	16	
240°	u_{16}	181	14	
255°	u_{17}	179	12	
270°	u_{18}	179	12	
285°	u_{19}	185	18	
300°	u_{20}	18[illegible]	15	
315°	u_{21}	176	9	
330°	u_{22}	166	−1	
345°	u_{23}	160	−7	

(u_0 to u_{12})	−18	−30	−39	−41	−39	−32	−8	11	22	24	22	20	11
(u_{23} to u_{13})		−7	−1	9	15	18	12	12	14	16	10	3	
Sums (v_0 to v_{12})	−18	−37	−40	−32	−24	−14	4	23	36	40	32	23	11
Diffs. (w_1 to w_{11})		−23	−38	−50	−54	−5[illegible]	−20	−1	8	8	12	17	

(v_0 to v_6)	−18	−37	−40	−32	−24	−14	4
(v_{12} to v_7)	11	23	32	40	36	23	
(p_0 to p_6)	−7	−14	−8	8	12	9	4
(q_0 to q_5)	−29	−60	−72	−72	60	−37	

(w_1 to w_6)	−23	−38	−50	−54	−50	−20
(w_{11} to w_7)	17	12	8	8	−1	
(r_1 to r_6)	−6	−26	−42	−46	−51	−20
(s_1 to s_5)	−40	−50	−58	−62	−49	

(s_1 to s_3)	−40	−50	−58
(s_5, s_4)	−49	−62	
Sums (k_1 to k_3)	−89	−112	−58
Diffs. (n_1, n_2)	9	12	

(p_0 to p_3)	−7	−14	−8	8
(p_6 to p_4)	4	9	12	
Sums (l_0 to l_3)	−3	−5	4	8
Diffs. (m_0 to m_2)	−11	−23	−20	

	n		*k*			*m*			*l*			
	9	12		−112			−23					
			−89		−89	−20		20	−4	−5	4	−5
$\frac{1}{2}$ × previous line .			−44·5			−10			−2	−2·5		
			−58		−58	−11		−11	−3	−8	−3	8
·866 × first line .	7·8	10·4		−97			−19·9					
Sum of 2nd column .	10·4		−97			−19·9			−10·5		3	
·707 × previous line .												
Sum of 1st column .	7·8		−102·5			−21			−5		1	
Sum	$18·2=12b_4$		$-199·5=12b_2$		$-31=12b_6$	$-40·9=12a_2$		$9=12a_6$	$-15·5=12a_4$		$4=24a_0$	
Diff. .	$2·6=-12b_8$		$5·5=-12b_{10}$			$1·1=-12a_{10}$			$-5·5=-12a_8$		$2=-24a_{12}$	

	q						*r*					
	−60						-51_5					
	−37						-6_1					
	−72	−97			72	−97	−46	−57			46	−57
	−60	−23	60	−23	−60	23	−26	−45	26	45	−26	45
	−30	−11·5			−30	11·5	−13	−22·5			13	22·5
	−29	−72	−29	72	−29	72	−20	−42	20	−42	−20	42
	−62·4	−84			62·4	−84	−39·8	−49·4			39·8	−49·4
	−167·5		49		−0·5		−113·9		3		15·1	
	−118·4		34·6		−0·4		−80·5		2·1		10·7	
	121·4		31		3·4		−72·8		−6		6·8	
	$-239·8=12a_1$		$65·6=12a_3$		$3·0=12a_5$		$-153·3=12b_1$		$-3·9=12b_3$		$17·5=12b_5$	
	$3=-12a_{11}$		$3·6=-12a_9$		$-3·8=-12a_7$		$-7·7=12b_{11}$		$8·1=12b_9$		$3·9=12b_7$	

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(0)
a	−20	−3·4	5·5	−1·3	0·3	0·8	0·3	0·5	−0·3	−0·1	−0·3	−0·1	0·2
b	−12·8	−16·6	−0·32	1·5	1·5	−2·6	0·3	−0·2	0·7	−0·5	−0·6		+167
a^2	400	11·6	30·3	1·7									
b^2	163·8	275·6	0·10	2·3									
$+b^2$	563·8	287·2	30·4	4·0									
$b^2)^{\frac{1}{2}}$	23·7	16·9	5·5	2·0									
$1/b$	−0·078	−0·60	−3·1	0·67									
a/b	1·56	2·04	−17·0	−0·87									
a/b	237·4	191·5	93	319									

The above example, which represents the diurnal variation of atmospheric electric potential gradient at Edinburgh during the year 1912, is the same as that worked by the twelve-ordinate scheme of Runge. S. P. Thompson [1] has devised computing forms which facilitate the analysis of a periodic curve in which only odd harmonics appear up to the fifth or eleventh order respectively. These are especially important in the case of alternating currents and electromotive forces in which the even harmonics are absent, and hence, if the base-line be so chosen that the mean ordinate be zero, the first and second half periods are similar, the signs of the ordinates in the second half being reversed. He has adapted the elaborate analysis of Runge to the case under consideration, and the following schedule is for the analysis of a periodic curve in which only odd harmonics appear up to the fifth order. The half period is divided into 6 equal parts, and the 5 ordinates u_1, u_2, u_3, u_4, u_5 are measured, u_0 and u_6 being zero. These are arranged as follows :—

	u_1	u_2	u_3
	u_5	u_4	
Adding . .	v_1	v_2	v_3
Subtracting .	w_1	w_2	

Denoting $v_1 - v_3$ by p_1, the form is as follows, each number before being entered being multiplied by the sine of the angle set opposite it :—

	Sine Terms.		Cosine Terms.	
Sines of angles .	1st, 5th	3rd	1st, 5th	3rd
Sin 30° = 0·500	v_1		w_2	
Sin 60° = 0·866	v_2		w'_1	
Sin 90° = 1·000 .	v_3	p_1		$-w_2$
Sum of 1st column . .	...	...	...	...
Sum of 2nd column . .	...	...	...	...
Sum . . .	$3b_1$	$3b_3$	$3a_1$	$3a_3$
Difference . . .	$3b_5$		$3a_5$	

[1] *Proc. Phys. Soc.*, xix. 443–450, 1905; *The Electrician*, 5th May 1905.

The result is—

$$u = a_1 \cos \theta + a_3 \cos 3\theta + a_5 \cos 5\theta$$
$$+ b_1 \sin \theta + b_3 \sin 3\theta + b_5 \sin 5\theta.$$

The following checks are applied :—

$$a_1 + a_3 + a_5 = 0 \qquad b_1 - b_3 + a_5 = u_3.$$

His second schedule, which gives a form for the analysis of a periodic curve in which only odd harmonics appear up to the eleventh order, will be found in his memoirs already referred to. He has also a schedule, which enables the odd harmonics up to the seventeenth to be calculated.

More recently the same writer [1] has explained another method of approximate harmonic analysis by selected ordinates. In this method the multiplication by sines or cosines is dispensed with, and the process simply consists in the arithmetical averaging of selected ordinates in addition to certain operations of addition and subtraction. The basis of the method, as stated by Thompson, lies in the easily verified fact that, " if a series of $2n$ ordinates is measured at intervals apart of π/n where n is the numeric representing the order of the harmonic, and if their values, *taken alternately positively and negatively*, are averaged over a whole period, the mean so obtained is either simply the amplitude of that harmonic or else is the sum of the amplitudes of that harmonic and of those of certain higher harmonics—namely, those the ordinal numeric of which is an odd multiple of n. For cosine components the series of $2n$ ordinates must begin (or end) at the beginning (or end) of the period. For sine components the series must begin at $\frac{\pi}{2n}$ from the beginning of the period." As distinct from his other methods, even and odd harmonics are dealt with here. In this method there is the limitation that in the calculation a higher harmonic may interfere with a lower one if the ordinal number of the higher is an odd multiple of that of the lower. This requires that these higher harmonics be either absent or separately evaluated and so taken account of. This method can be applied to the case of periodic phenomena such as the tides, diurnal variations in meteorological phenomena, and valve gear motions, and he has drawn out special forms for dealing with these phenomena. For example, one of his schedules suitable for harmonic analysis of valve motions, etc., enables us to find the first three harmonics, those above that order being assumed to be absent. The problem is to find a_1, a_2, a_3, b_1, b_2, b_3. The period being 2π, ordinates are read off at intervals of 30° beginning at 0°; then

$$\begin{aligned}
a_3 &= \tfrac{1}{6}(u_{0°} - u_{60°} + u_{120°} - u_{180°} + u_{240°} - u_{300°}) \\
b_3 &= \tfrac{1}{6}(u_{30°} - u_{90°} + u_{150°} - u_{210°} + u_{270°} - u_{330°}) \\
a_2 &= \tfrac{1}{4}(u_{0°} - u_{90°} + u_{180°} - u_{270°}) \\
b_2 &= \tfrac{1}{4}(u_{45°} - u_{135°} + u_{225°} - u_{315°}) \\
a_1 &= \tfrac{1}{2}(u_{0°} - u_{180°}) - a_3 \\
b_1 &= \tfrac{1}{2}(u_{90°} - u_{270°}) + b_3 \\
a_0 &= u_0 - a_1 - a_2 - a_3.
\end{aligned}$$

In addition, he has drawn out schedules for analysing curves involving harmonics up to the seventh order, higher harmonics being assumed absent,

[1] *Proc. Phys. Soc.*, xxxiii. 334–343, 1911 ; *Arkiv för Mathematik, Astronomi o. Fysik*, Bd. 7, No. 20. See also Fischer-Hinnen, *Elektrotech. Zeit.*, xxii. 396, 1901.

schedule suitable for curves involving only odd harmonics up to the ninth order, and, finally, a special schedule suitable for the analysis of tidal observations.

Other arithmetical methods are due to Perry [1] and Kintner,[2] who has extended Perry's method. The method consists in measuring off equidistant ordinates and multiplying the values through by the appropriate value of $\sin n\theta$, the results being tabulated and averaged for each harmonic. H. H. Turner [3] has recently published tables for facilitating the use of harmonic analysis. The tables are arranged so that the values of $a_{r+1} \begin{smallmatrix}\cos\\ \cdot \\ \sin\end{smallmatrix} r\theta$ may be got to two figures, and are useful in connection with Schuster's periodogram method.

III. Graphical Methods

A very large number of graphical methods have been devised, but naturally they are not so accurate as the arithmetical ones. They are useful, however, in many cases, particularly when only the first few harmonics are required and when expert computers are not employed. A few of these methods will be briefly described here, and references given for a number of others.

Wedmore's method [4] enables the amplitude and phase of the successive harmonics to be determined. If a period of the curve to be analysed be divided into two portions by an ordinate bisecting the base-line, then on superposing these portions all the ordinates of the harmonics whose frequencies are not multiples of two annul, while the ordinates corresponding to frequencies which are multiples of two are added. Hence, if the ordinates of the resulting curve be divided by two, we have a curve in which those harmonics of the original curve are present whose frequencies are multiples of 2. For instance, if only the first 4 harmonics be present, then by repeating the above process again the amplitude and phase of the 4th harmonic is got. By dividing the original curve into three portions instead of two, the 3rd component will be determined. The curves require to be carefully drawn, and the accuracy is increased by drawing on a large scale.

In Perry's[5] graphical method, based on the graphical method of Clifford already referred to, we suppose that n values of the function $f(x)$ are known, and a circle is drawn with radius $\frac{n}{2\pi}$. The circumference is divided into n equal parts, and the projections of these points are got on a horizontal diameter. The points on the diameter are numbered 0, 1, 2 . . . n, while the points on a perpendicular line, which are also numbered 0, 1, 2 . . . n, are got by measuring along this perpendicular from its intersection with the horizontal line distances proportional to the values of $f(x)$ for $x=0$, 1 . . . respectively; the intersections of lines drawn

[1] *The Electrician*, xxviii. 362, 1892.

[2] *Electrical World and Engineer*, xliii. 1023, 1904.

[3] *Tables*, etc., Oxford University Press, 1913.

[4] *The Electrician*, 1895; *Jour. Inst. Elect. Eng.*, xxv. 234, 1896; Kelsey's *Physical Determinations*, p. 90, 1907.

[5] *The Electrician*, xxxv. 285, 1895; Kelsey's *Physical Determinations*, p. 86.

perpendicular and parallel to the horizontal diameter through the points corresponding to $x=0$, $f(x)=0$, etc., will give a curve whose area divided by $\frac{n}{2}$ gives the coefficient a, etc. The complete scheme is given in the memoir referred to.

R. Beattie [1] has described a graphic method in which special scales are used. For example, to find a_n (in $a_n \cos n\theta$) a reciprocal-cosine scale would be used, and the period of the curve to be analysed having been made equal to the base-line of the scale, the scale is placed so that the base of the curve coincides with the base-line of the scale. On this scale there are drawn a number of vertical lines at distances representing θ_1, θ_2 . . . and the lines are divided into scales whose units are $1/\cos n\theta_1$, $1/\cos n\theta_2$, etc., the zero of the scales being on the base-line.

If m be the number of ordinates selected, then

$$a_n = \frac{2}{m}(z_1 + z_2 + \ . \ . \ .),$$

where $z_1 = y_1 \cos n\theta_1$, $z_2 = y_2 \cos n\theta_2$, etc., and are read directly from the scales. Similarly for b_n. Details of scales, etc., are given in the original paper. Beattie [2] has also published an extension of Fischer-Hinnen's method of harmonic analysis, and has shown how scales similar to the specially graduated scales which he has designed for his method (*loc. cit.*) can be adapted for use with the Fischer-Hinnen method.

Harrison's [3] method consists in drawing the ordinates of the curve to be analysed as vectors at equal angles from a given point, and by projection on the two rectangular axes the amplitude and phase of a harmonic can be got. Ashworth [4] modifies this method and treats the ordinates as coplanar forces radiating from a common centre at angles 2θ, $2n\theta$, etc. The resultant of these can be found by the polygon of forces, and gives the amplitude, while the phase can be found by measuring the angle which this resultant makes with the x-axis.

References to various methods will be found in Burkhardt's article in the *Encyk. d. math. Wissenschaften*, Bd. ii. Th. i. 642, 1904; Beattie's article, *The Electrician*, lxvii. 326, 1911; Darwin, *Engineering*, p. 81, 1911; Pichelmayer and Schrutka, *Elektrotech. Zeit.*, xxxiii. 129, 1912; F. Meurer, *Elektrotech. Zeit.*, xxxiv. 121, 1913; H. Rottenburg, *The Electrician*, lxx. 1140, 1913; S. Silbermann, *Elektrotech. Zeit.*, xxxiv. 936, 1913; R. Slaby, *Arch. f. Elektrotech.*, p. 19, 1913.

[1] *The Electrician*, lxvii. 326, 370, 1911.
[2] *Ibid.*, lxvii. 847, 1911.
[3] *Engineering*, lxxxi. 201, 1906.
[4] *The Electrician*, lxvii. 888, 1911.

VII. Tide-predicting Machine. By Edward Roberts, F.R.A.S.

The accurate prediction of the tides is a matter of very great importance to maritime nations, more especially to those whose shores are subject to a considerable tidal action.

It is well known that the fluctuations of the sea may be expressed by a series of cosines of multiples of the times when the periods are known; but it was not until the subject of the reduction of tidal observations by the method of harmonic analysis was taken up by a committee of the British Association in 1867, and continued for some years under the chairmanship of Sir Wm. Thomson (Lord Kelvin), that tidal constants were determined in a suitable form.

In the machine there are parts or movements for representing the mean action due to the sun and moon, and similar movements correct for the ellipticity of the lunar orbit and also for the moon's motion out of the equator. In the case of the sun one such movement is included for the ellipticity of the earth's orbit, but two, as in the case of the moon, for the sun's motion in the ecliptic. Other movements are necessary in the case of the moon, those correcting for the ellipticity of its orbit not being sufficiently accurate to fully represent the orbit; the next two largest inequalities, termed the *evection* and *variation*, have therefore been included.

Other similar movements correct for the effect of friction, a number of these movements being necessary to represent accurately the tides of rivers and seaports with a shallow foreshore. In addition to the above, other movements again correct for the effects of temperature and rainfall, which must be included to predict with all practical accuracy the tides at any port.

The number of tide-components that can be combined on the machine is forty. Some of these, however, are not actually geared up, but may be included if tidal analysis shows them to be desirable.

The movements are fitted on a metal plate measuring about 6 feet by 3 feet, in an upper and a lower series. The upper series contains 21, and the lower 19 components. For each component there is a pulley fitted on a parallel slide, actuated by a pin fitted on a crank turning in its proper period relatively to the other components. It is counterbalanced, to avoid wear and friction on the crank-pin. The crank-pins are set to scale to their proper values as determined from the actual reduction of the tidal observations of the port for which the predictions are required. The time of actual maximum of each component is likewise found from the observations. The crank-pin moves in a slot in the horizontal bar of the parallel slide. The axis of each crank is fitted with a slotted cone to enable it to be freed and adjusted to its proper position at starting. The setting dials are carried on two plates at the back, and the wheelwork actuating the whole is between these and the main front plate.

The main plates are supported on standards nearly 3½ feet high. Between the standards are fitted, in the centre the recording drum, and at either side a drum with a supply of continuous paper and a haul-off drum receiving the paper after tracing by the recording pen.

A fine flexible wire, attached to a screw-head fitted near the centre of the

main plate, passes under and over the pulleys of the components of the right-hand lower section, and then passes similarly over and under the upper section of components from right to left, and then under and over the left-hand lower section, finally leaving the pulley of the main lunar semidiurnal component near the centre of the plate. From this, the free end of the wire, is suspended a recording pen fitted with a fine glass point and carrying an ink reservoir. The pen-carrier runs in a vertical slide, and is suspended so as to give just sufficient pressure to ensure contact with the paper on the recording drum. The recording drum is fitted with brass pins at equal distances, which by perforations mark the hourly positions of the record—noon of each day is indicated by a double perforation. The travel of paper generally used is 6 inches to the day, or one-quarter inch per hour. Pens for tracing the mean tide level or datum level are fitted on an upright bar near the pen slide. The depth of paper on the recording drum is 29 inches.

A date dial is provided to enable the record to be marked occasionally to facilitate the measurements for time and height after the record has been removed from the machine.

The machine is driven by a small electric motor, and a year's tracings for any port are run off in about two hours.

(1) Exhibit and Demonstration of the Roberts Tide-predicting Machine

(2) Lord Kelvin's Tide-Predicter. Photograph. Lent by Messrs Kelvin, Bottomley, and Baird

The Tide Predicter is a machine which performs the operation of adding together a series of harmonic tidal components, the resultant tide being drawn as a continuous curve or graph on a paper chart ; or, in symbols, it draws the graph—

$$y = \mathrm{A} \cos (\alpha t + \lambda) + \mathrm{B} \cos (\beta + \mu t) + \ldots$$

by performing a mechanical summation of the constituent terms.

We must suppose that the constants of all the tidal constituents have been determined by harmonic analysis of the tide gauge records for the port in question. That is to say, the amplitude of each harmonic constituent and its phase relationship with all the others are known. The Tide Predicter, then, is a mechanism which generates a number of simple harmonic motions similar in all respects to the corresponding tidal motions ; these motions are further added together algebraically at every instant, and the resultant motion is recorded continuously on a paper chart. Afterwards, from this chart the heights and times of high and low water can be taken and reduced to the usual tabular form.

Turning to the illustration of fig. 1, a number of pairs of toothed wheels will be seen, the lower member of each pair being carried on a horizontal shaft common to all. Each tidal component to be included in the prediction has such a pair of wheels allotted to it, and the numbers of the teeth on the

wheels are chosen so that if one revolution of the common shaft corresponds to one day, then the number of revolutions made by the upper wheel is a close approximation to the true frequency of this component.

Each pair of wheels has a large pulley above it, and the rotation of the upper wheel is arranged to give a harmonic up-and-down motion to the pulley over it by means of a pin-and-slot mechanism, to be seen to the right of the toothed wheel. The slotted link, to which the pulley is attached by a light

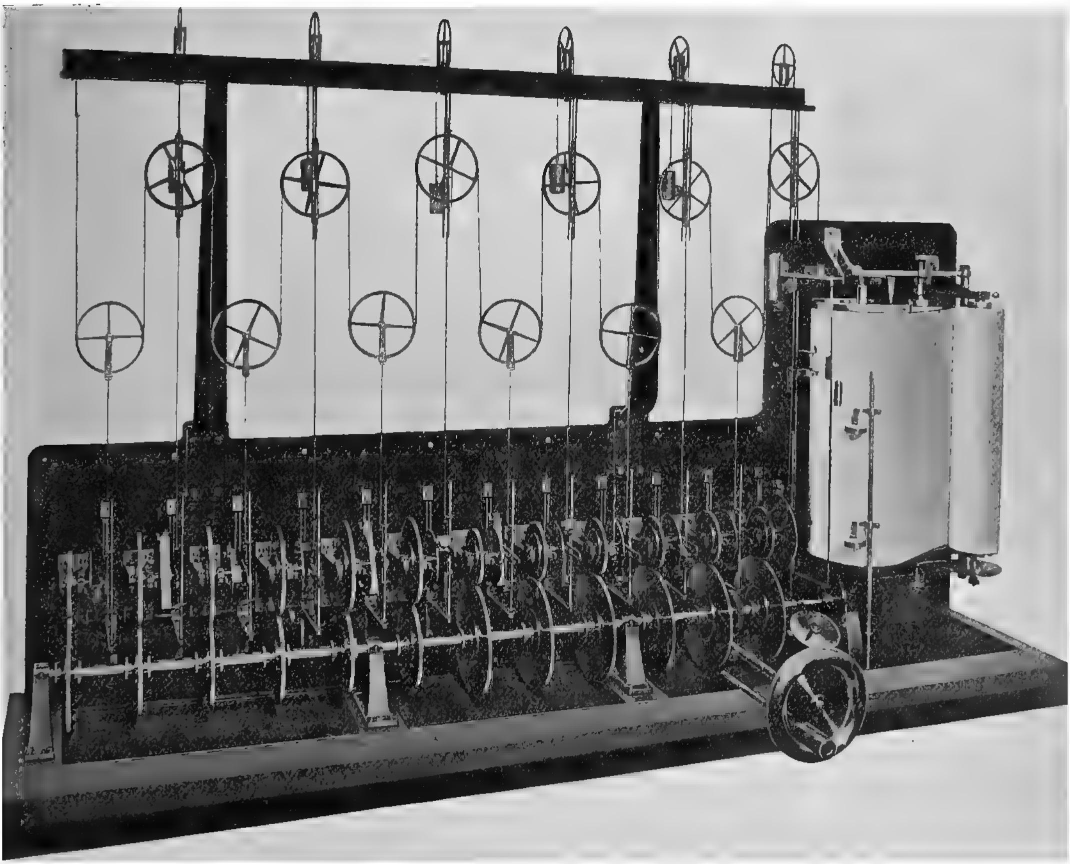


Fig. 1.

rod, is constrained by guides to move vertically. Consequently the pulley is moved up and down as the pin revolves with its wheel and moves the link.

The pulleys are placed alternately high and low, and a continuous fine wire passes under and over them. The wire is fixed at its left extremity to an adjustable screw in the frame of the machine, and ends on the right at a pen which moves vertically over the surface of a drum round which a chart paper is fed.

As the pulleys rise and fall the vertical portions of the wire are lengthened or shortened and the pen is caused to move up or down, tracing a record on

the paper. It is clear that the vertical motion of the pen will be twice the algebraic sum of the motions of all the pulleys.

VIII. A Mechanical Aid in Periodogram Work

Exhibited by D. Gibb, M.A.

In the discussion of any sequence of observations, such as the brightness of a star or the temperature at any station on consecutive days, in which periodicity is suspected, it is not always possible to determine the periods graphically. When this is so, recourse must be had to arithmetical processes.

The method is as follows :—Let m_n denote the observation on the nth day, counting from the beginning of the observations. Then, in order to test whether a periodicity of (say) fifty days exists, the observations are written down thus :—

$$\begin{array}{cccccccc} m_1 & m_2 & m_3 & \cdot\;\cdot\;\cdot & \cdot & & m_{49} & m_{50} \\ m_{51} & m_{52} & m_{53} & & \cdot & \cdot & m_{99} & m_{100} \\ m_{101} & m_{102} & m_{103} & & \cdot & & m_{149} & m_{150} \\ \cdot\;\cdot & \cdot\;\cdot & \cdot\;\cdot & \cdot\;\cdot\;\cdot & \cdot & \cdot & \cdot & \cdot\;\cdot \end{array}$$

Each horizontal row contains fifty consecutive observations, and is called a "lap." We take a convenient number of laps—the more the better—and then sum the numbers that stand in each vertical column of the scheme. Let the sums be denoted by

$$M_1 \quad M_2 \quad M_3 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad M_{50}$$

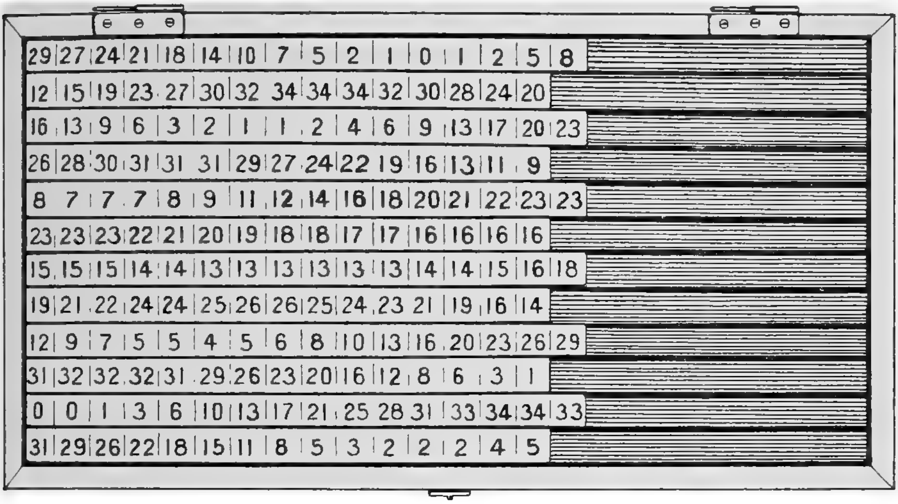
Then if there are k laps, the periodicity of fifty days will be intensified k-fold in the sequence $M_1, M_2, \ldots M_{50}$, as compared with its intensity in any one of the horizontal rows, for this periodicity enters with the same phase into every horizontal row, and its amplitude will therefore be k times as great in the sum of k rows as in a single row. On the other hand, periodicities other than the fifty-day periodicity will occur with different phases in each horizontal row, and when the vertical columns are summed the elements with different phases will annul each other, so that these periodicities will not appear in the sequence $M_1, M_2, M_3, \ldots M_{50}$.

Thus we have obtained a sequence $M_1, M_2, \ldots M_{50}$, in which the fifty-day periodicity, if it exists, is greatly intensified, while the other periodicities are destroyed. The difference between the greatest and the least of the numbers $M_1, M_2, \ldots M_{50}$ therefore furnishes a rough indication of the amplitude of the fifty-day periodicity.

The writing down of the observations in laps corresponding to the different trial periods is exceedingly laborious, and the writer has devised the following means of avoiding it :—

Each observation is marked on a small wooden cube, and the cubes are arranged in rows in a wooden frame, just as in the above scheme. The advantage of using the cubes is that when a change has to be made from one

trial periodicity (say fifty days) to another trial periodicity (say fifty-one days), the change is effected by simply sliding the cubes along in their rows and transferring a few cubes from the beginning of each row to the end of the row above it: no rewriting is needed.



MECHANICAL AID IN PERIODOGRAM WORK

FIG. 1.

By the aid of this device, and with a comptometer to add the numbers in the vertical columns, the search for periodicities can be carried out with much greater rapidity than has been hitherto attained.

IX. The Mechanical Description of Conics. By D. GIBB, M.A.

THOUGH conography or the mechanical description of conics has attracted the attention of mathematicians for many centuries, it cannot be said to have found favour with those by whom these curves are constantly used. This may perhaps be due to the circumstance that the instruments are somewhat cumbersome, and can usually describe only a small portion of the curve. Of the numerous mechanisms which have been invented, probably only two—the ellipsograph of Proclus, and that for describing the "gardener's curve"—are ever employed. Even engineers, who are constantly making use of stress and strain ellipses in the theory of the strength of materials, and in the theory of elasticity, and of parabolæ in the theory of bending moments, prefer either to draw the curves directly from their equations, or to use a simple graphical method of construction. Many of these instruments, however, give very accurate representations of portions of conics, satisfying given conditions, and, on that account, are worthy of the attention of the users of these curves.

Probably the many fruitless attempts made by the ancients to solve the

Delian problem gave rise to the construction of conics and higher plane curves. Plato, who condemned the organic description of geometrical figures as tending to materialise geometry and to bring it down from the region of eternal and incorporeal ideas, is said to have solved this problem by means of an instrument, a diagram of which has been given by Eutokius of Ascalon. This was the first instrument for solving a geometrical problem, and, on that account, is worthy of mention here, though it was not employed for the description of a curve. Not content with this empirical solution of the Delian problem, Plato's school sought for new means to overcome the difficulty, and one, Menæchmus, discovered the conic sections. Utilising these, he solved the famous problem, first of all by means of two parabolæ, and then by means of a parabola and a hyperbola. So we can scarcely err if we assign the invention of the first conograph to the time of Menæchmus. Indeed, Eratosthenes mentions that Menæchmus had used instruments for the construction of his curves, but of what kind he does not say.

In the meagre account which comes to us in the later works of the Grecian geometers, we find an interesting note in the commentary which Proclus (410–485 A.D.), the chief of the Platonic school at Athens, wrote to Euclid's works. This gives the mechanical construction of an ellipse as the motion of a point P (fig. 1) on a straight line AB, whose extremities describe two fixed straight lines OX, OY. Considering the practical application which the Greeks gave to their discoveries, we may be sure that Proclus' idea was actually put into practice. Thus we may ascribe to him the discovery of the principles on which are now based innumerable instruments which differ only in technical construction. Following the ellipsograph of Proclus (fig. 1) was the discovery of an instrument for the mechanical description of parabolæ by Isidorus of Miletus, who likewise applied it to the solution of the Delian problem. In his account Eutokius only mentions that the instrument had the form of the Greek letter λ. Such is the trifling share we receive from the Greeks.

In Arabic literature the three famous problems of the Greeks again come into prominence, and many different solutions of them are obtained. To these must be added the solution of equations of the third and fourth degrees by means of conic sections. As regards the instrumental description of curves, we next find the so-called " gardener's construction " of the ellipse by means of a pencil which keeps taut a thread whose extremities are kept at two fixed points. This was discovered by Alhasan, the youngest son of Musa Jbn Schaker, an influential personage at the court of the Caliph Al-Mamum (813–833). Again, in the last of three treatises which Franz Wöpcke has handed down to us is shown an instrument invented by the Arabs for the description of conics. This mechanism owes its formation to the early observation by the Arabs that the extremities of the shadow of a gnomon lie on conic sections. The curve is simply a plane section of a cone. This instrument is very similar to that invented in 1566 by Barocius (fig. 2). The latter consists of an axis AB, which can be set at any angle with the plane on which the instrument is fixed, and which can be lengthened or shortened by means of a movable piece BC. To the top of the latter is attached a tube DE, which can be inclined at any angle to the axis. The pencil at E must

fit the tube so loosely that, when it is rotated about the axis in order to describe the curve, it shall always be in contact with the paper.

Another instrument (fig. 3) which depends on the same principle as that of Barocius was constructed by Christoph Scheiner (1573–1650). This consists of an axis AB, which can be set at any angle with the plane KLMN; a graduated semi-circle, which is easily movable above the axis, and which can be fixed in any position on the same by means of the cones C and E; and a bar FG, which can be moved upwards and downwards on the screws I and D, the latter of which serves to keep it inclined at a definite angle to the axis. In this case also the bar FG must move so freely that the pencil G will remain in contact with the paper during the rotation of the semi-circle about AB. Scheiner's pupil, Georg Schönberger, who describes the instrument, claims that it can describe straight lines, circles, and the three conic sections. Straight lines, he says, are obtained if the axis is inclined to the paper, and the pencil is at right angles to the axis; circles if the axis is perpendicular to KLMN, and the pencil at an acute angle with it. If the axis is inclined at an angle $QAI=45^\circ$ with the plane, an ellipse is obtained if the angle $AID<45^\circ$; a parabola if $A\hat{I}D=45^\circ$; and a hyperbola if $>45^\circ$ but $\lesseqgtr 90^\circ$. If $A\hat{I}D<90^\circ$, then the hyperbola faces towards A, but if $A\hat{I}D>90^\circ$, then it faces in a direction perpendicular to this. This instrument shows particularly well the genesis of the conic as the section of a cone. For the construction of sundials, for which it was invented, this instrument may have been useful, but, like that of Barocius, it would not satisfy the present-day demands for accurate drawing.

These seem to have been forgotten for a number of years, for in 1684 the same idea, in another but more complicated form, is again put into practice by Benjamin Bramer. His instrument (fig. 4) resembles most that of Barocius in that the pencil CD moves in a tube CE, and the plane AB can be inclined at any angle to the axis GH. The rotation of the tube is effected by means of the key I. This apparatus may possibly give better curves, but as it requires a massive stand as well as an arrangement for fixing the drawing board, it is less convenient. When the drawing board has the position shown in the figure, a parabola is obtained; an ellipse if it is tilted upwards; a hyperbola if downwards.

To these may be added an instrument which shows how to construct a hyperbola whose foci and the constant difference of whose focal radii are given. A description of this, which corresponds to the gardener's construction of an ellipse, is found in a manuscript of the famous Italian, Guido Ubaldo del Monte (1545–1607). Nor must the influence exerted by the famous mathematician, Reni Descartes, be overlooked. Though the mechanisms which he himself invented were chiefly for the construction of higher plane curves, his followers, especially Franz von Schooten, devoted much of their time to the construction of conographs. The latter, who spread the idea both in his writings and in his teaching, constructed many instruments depending on the properties of the ellipse. For instance, he showed that every point of a plane figure invariably connected with the line AB in fig. 1 describes an ellipse, and that if a line AB of length l moves in such a way that one of its extremities A describes a circle C of radius l, and

the other B a diameter of this circle, then every point Q of the plane invariably connected with AB describes an ellipse.

The most important additions in the eighteenth century, which might also be considered the precursors of the newer system of projective geometry discovered by J. Steiner and further developed by M. Chasles, were those of Newton and Maclaurin. In Newton's case the apparatus is based on the theorem that if two angles of given magnitude turn about their vertices in such a way that the point of intersection of one pair of arms lies always on a fixed straight line, then the point of intersection of the other pair of arms will describe a conic. Maclaurin's method, which was also discovered independently by Braikenridge, is really a generalisation of the above. It depends on the theorem that if a variable polygon move in such a way that its n sides turn severally round n fixed points, while $n-1$ of its vertices slide respectively along $n-1$ fixed straight lines, then the last vertex will describe a conic. Fig. 8 shows a particular case of this. The sides of the triangle OAB rotate about the fixed points P, Q, R, while the vertices O, B describe the fixed straight lines XY, XZ. The point A then describes a conic, passing through the five points P, Q, X, Y, Z, so that the conic is unique.

It was natural that with the further development of projective geometry, which lends itself to easy geometrical constructions, other methods of generating conics should arise. Such, for example, is the conograph (fig. 7) invented by Willy Jürges,[1] in which four bars, having grooves on their lower sides, turn about a point. The four pins, 1, 2, 3, 4, are set on four fixed points, and then after setting the vertex on a fifth fixed point, the bars are so adjusted that the heads of the four pins fit into the four grooves. The four blocks, I, II, III, IV, which are movable above a vertical axis, are then slipped on the bars and the transversal firmly fixed on them. By passing a pencil through the hollow cylinder at the vertex a conic may be traced through the five fixed points. By means of this apparatus we can describe conics to satisfy various conditions; for instance, to pass through three given points and to touch a given line at a fixed point. It may also be used for the construction of tangents at given points on the conic.

In conclusion, we may briefly describe the remaining mechanisms shown in the plate.

Fig. 5 is W. Rottsieper's conograph[2] for the description of a hyperbola whose asymptotes are given. This depends on the property that the portions of a chord intercepted between a hyperbola and its asymptotes are equal. It follows that the projections of these on the x-axis, parallel to the y-axis, are equal. The slotted bar RQ is therefore fixed on the waggon so that it shall be parallel to the y-axis. The bar SP moves about a pivot at S, and through a fixed point P. By placing a pencil in the hollow pulley at Q, and moving the waggon parallel to the x-axis, a hyperbola is described, having OX, OY as asymptotes.

Fig. 6 is Cunynghame's hyberbolograph.[3] This depends on the property that if O is a fixed point and PQ be drawn perpendicular to a fixed line, and

[1] *Zeitschrift für Mathematik und Physik*, xxxviii. (1893), p. 350.
[2] *Ibid.*, lxi. (1913), p. 74.
[3] *Philosophical Magazine*, 5th series, vol. xxii. p. 138.

if the sum of OP and PQ is constant, then the locus of Q is a rectangular hyperbola. The method of using the mechanism is obvious.

Figs. 9 and 10 are really linkages for the construction of conics. The former is Burstow's ellipsograph.[1] ODC is a fixed straight line, and O a fixed point. OA, AC are links, the extremity C of the latter being constrained to slide along the line OC. At B, the middle point of AC, a link BD, of length equal to AB, is jointed, and D is made to move along OC. If DE be kept parallel to OA, then E will describe an ellipse. The latter is one of the instruments designed by Guest [2] for generating the whole of a conic. It makes use of Kempe's variation of the Hart cell to describe hyperbolæ referred to their asymptotes. In this mechanism, if LSM, MPK, KQN, and NOL be similar triangles described upon the bases LM, MK, KN, NL of the Hart contra-parallelogram, then OSPQ is a parallelogram of constant area. Hence, by fixing O and making Q slide on a straight line passing through O, the point S is forced to describe a straight line through O, and P to describe a hyperbola, of which the paths of S and Q are asymptotes.

Another method of constructing a hyperbolograph is as follows :—

Let AB and BC be two rods inclined to each other at any angle. Let P be a ring sliding on the rod BC and F a fixed point in the plane of ABC. If now a thread of length equal to BC pass through the ring P and have its extremities fixed at C and F, then P will trace out a hyperbola when AB moves along a line XX′. This line XX′ will be the directrix, F the focus, and BC the direction of an asymptote.

In the particular case in which the angle ABC is a right angle the apparatus becomes a parabolograph.

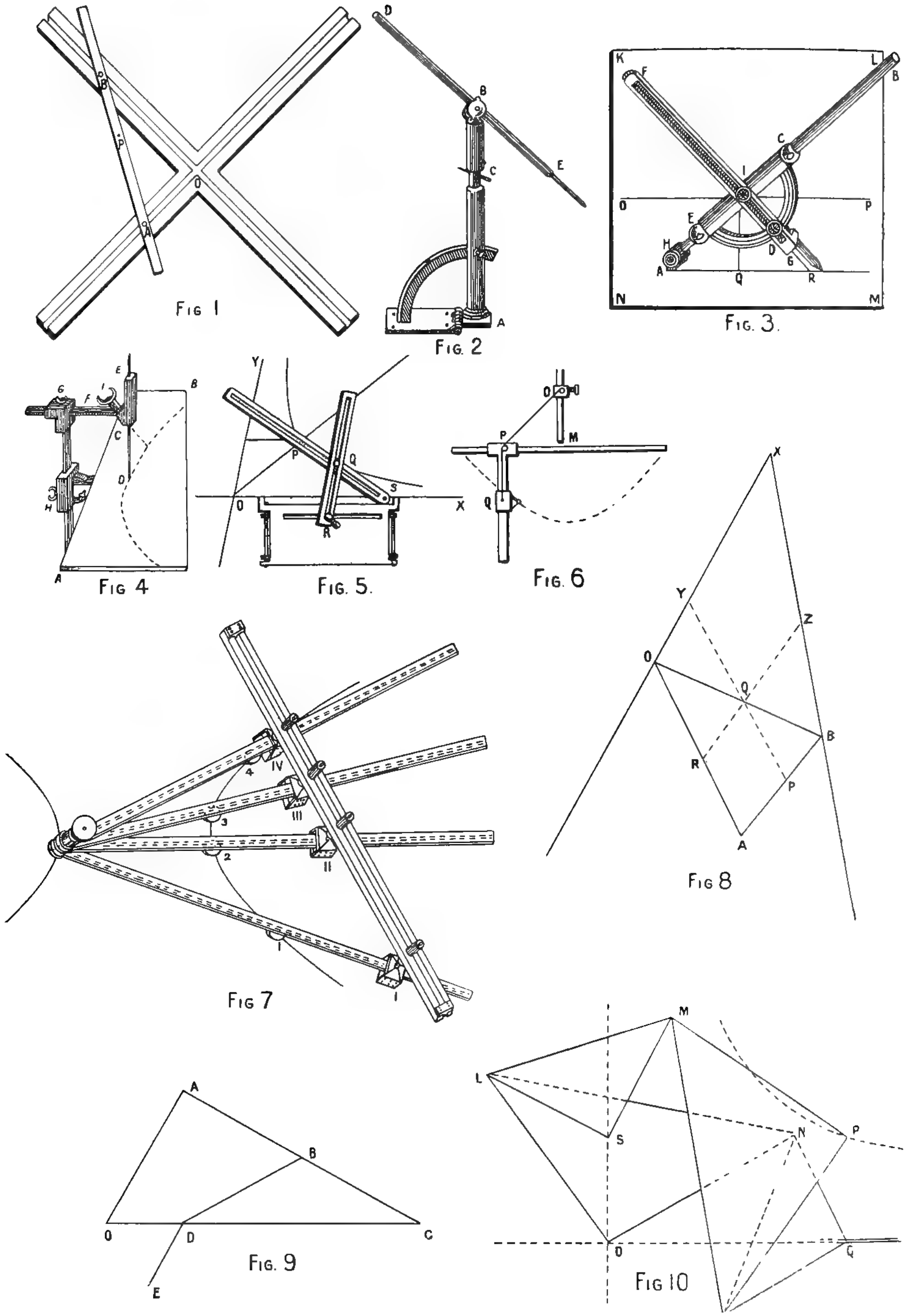
A beautiful method of generating a conic and its inverse at the same time is described by Sylvester.[3] It may be briefly described thus :—

Let Q, F, P be the three collinear points of a Peaucellier cell, taken in order. Let the point F instead of (as is usual) the point Q be fixed, and by introducing an extra link let Q describe an arc of a circle passing through F. The point P will then trace out a nodal cubic whose equation in polar coordinates is of the form

$$r = a \sec \theta - b \cos \theta.$$

But this is the inverse of a conic with respect to its vertex. Hence, by adding a second Peaucellier cell to invert the curve described by P, we can obtain a conic. If the conic to be described is a parabola, the curve traced by P will be the cissoid.

[1] Made by Stanley, London.
[2] *Proc. and Trans. of the Roy. Soc. of Canada*, 2nd series, vol. ii. sect. iii. p. 25.
[3] *Proc. Roy. Inst.*, vii. (1873–75), p. 179.



Group of Conographs exhibited by D. Gibb, M.A.

Fig. 1.—Ellipsograph of Proclus.
Fig. 2.—Barocius' Conograph.
Fig. 3.—Scheiner's ,,
Fig. 4.—Bramer's ,,
Fig. 5.—Rottsieper's ,,
Fig. 6.—Cunynghame's Conograph.
Fig. 7.—Jürge's ,,
Fig. 8.—Maclaurin's ,,
Fig. 9.—Burstow's ,,
Fig. 10.—Guest's ,,

X. The Instrumental Solution of Numerical Equations.

By D. Gibb, M.A.

The various methods of solving numerical equations may be classified as follows :—

(i) Solution by means of radicals.
(ii) ,, ,, series.
(iii) Arithmetical or computing method.
(iv) Graphical method.
(v) Instrumental method.

Of these the last only, the instrumental method, concerns us at present.

Mechanisms for the Solution of Equations with one Unknown

The invention of instruments or machines which will solve equations without any further calculation has a very great practical importance. Greek mathematicians knew the solution of the Delian problem, which

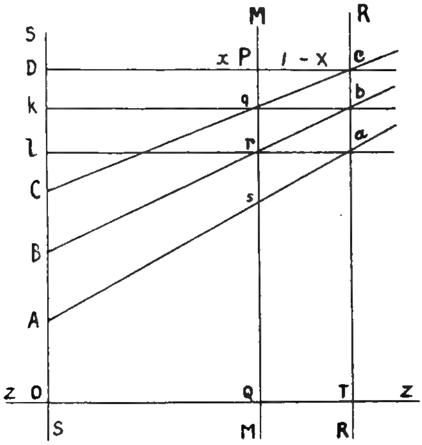


Fig. 1.

required the extraction of a cube root. The mechanical solution of this problem, attributed by Eutokius to Plato, may therefore be taken as the first instrumental solution of an equation. This solution depended on the use of two right angles, and is really the same solution as that obtained in the sixth century by means of the curve known as the "Cissoid of Diocles."

The mechanism invented in 1770 by J. Rowning[1] depends on the same principle as the method for the graphical representation of rational algebraic functions. The mechanism invented by Dr R. F. Muirhead (*q.v.*) depends also on the same principle. Rowning's is really an instrument which, by combinations of appropriate mechanism, permits of the tracing by a continuous

[1] *Phil. Trans.*, vol. lx. (1770), p. 240.

movement of certain curves of high order arising in the graphical solution. The principle of this instrument may be briefly described thus :—

Let the equation to be solved be

$$a+bx+cx^2+dx^3=0.$$

On ZZ as base draw perpendiculars SS, MM, RR at any convenient distances apart.

Set off OA, AB, BC, CD equal to the coefficients a, b, c, d. Through D draw Dc parallel to ZZ. Join cC, cutting MM in q. Draw kqb parallel to ZZ and join bB, cutting MM in r. Draw lra parallel to ZZ and join aA, cutting MM in s.

Let Dc be taken as unit length, and DP as equal to x. Then, since DCc and Pqc are similar triangles, we have

$$\mathrm{P}q : \mathrm{CD} = \mathrm{P}c : \mathrm{D}c$$
$$\therefore \quad \mathrm{P}q = d(1-x)$$

and

$$\therefore \quad k\mathrm{B} = \mathrm{BC} + \mathrm{CD} - k\mathrm{D}$$
$$= c + dx.$$

Similarly

$$kb : qb = k\mathrm{B} : qr$$
$$\therefore \quad qr = (1-x)\ (c+dx)$$

and

$$\therefore \quad \mathrm{A}l = \mathrm{AD} - \mathrm{D}k - kl$$
$$= b + cx + dx^2.$$

Again

$$la : ra = \mathrm{A}l : sr$$
$$\therefore \quad sr = (1-x)\ (b+cx+dx^2)$$

and

$$\therefore \quad \mathrm{Q}s = a+bx+cx^2+dx^3.$$

Consequently when Qs=0—that is, when the curve described by s, as MM moves parallel to SS or RR, cuts the base ZZ—we have $a+bx+cx^2+dx^3=0$. Hence the vaues of OQ or x, which render $a+bx+cx^2+dx^3$ zero, will render Qs zero. Thus the points at which the curve traced out by s cut the ZZ axis give the real roots of the equation. The method may be extended to equations of any degree whatever.

A. B. Kempe [1] has recourse to a jointed system. The equation to be solved is

$$u=a_0+a_1x+a_2x^2+\ \ldots\ +a_nx^n=0 \qquad . \qquad . \qquad (1)$$

First of all he obtains an upper limit a and a lower limit b to these roots. Then he chooses a quantity c equal to the numerically greater of a and b, and puts $x=c\cos\theta$. Substitution of this value in (1) gives

$$u=a_0+a_1c\cos\theta+a_2c^2\cos^2\theta+\ \ldots\ +a_nc^n\cos^n\theta=0,$$

which, by another well-known transformation, can be put in the form

$$u=c_0+c_1\cos\theta+c_2\cos 2\theta+\ \ldots\ +c_n\cos n\theta=0.$$

The equation is then in a form suitable to be dealt with by his machine.

[1] *Messenger of Mathematics* (1873), p. 51.

A series of levers AB, BC, . . . MN are jointed together, each being compelled by a simple mechanical means to make the same angle with its neighbour as AB does with AX.

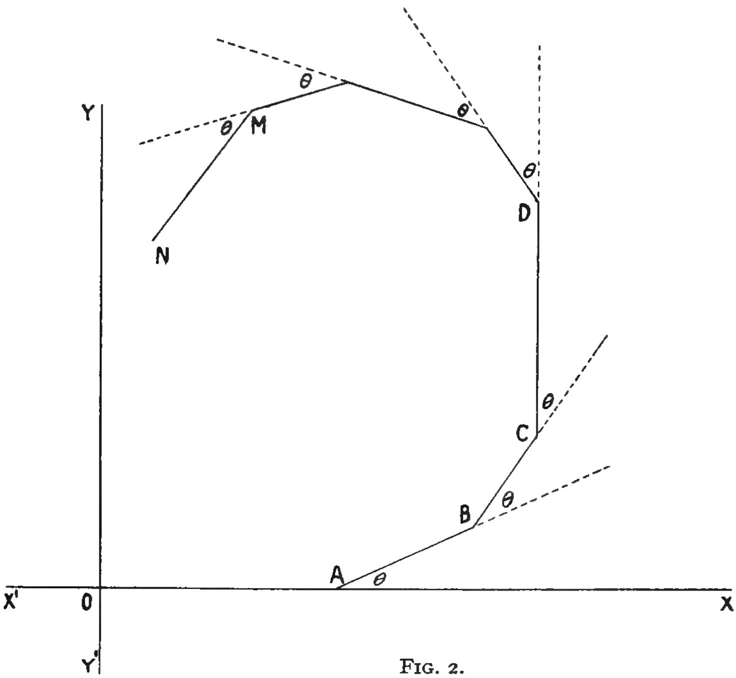


FIG. 2.

Let OA$=c_0$, AB$=c_1$, . . . MN$=c_n$. Then if AB makes an angle θ with AX, it is evident that the perpendicular distance of

A from YOY' is c_0
B ,, ,, ,, $c_0+c_1 \cos\theta$
C ,, ,, ,, $c_0+c_1 \cos\theta + c_2 \cos 2\theta$
. . . .
N ,, ,, ,, u.

Thus if AB revolve about A so that θ varies from 0 to π, and therefore x from $+c$ to $-c$, it is evident that when N lies on YOY', $u=0$, and the corresponding value of x or $c\cos\theta$ is a root of (1). The curve traced out by N will cut the y-axis as many times as there are real roots of the equation.

F. Bashforth [1] has described an instrument for the study of the more general form

$$c_0+c_1\cos(\theta+\alpha_1)+c_2\cos(2\theta+\alpha_2)+ \ . \ . \ . \ +c_n\cos(n\theta+\alpha_n).$$

It is claimed that this instrument may be employed to find the numerical roots of equations correct, probably, to two places of decimals. The accuracy of the values given would very nearly correspond to that of the ten-inch slide rule—the first two figures would be correct, the third doubtful.

[1] *Brit. Assoc. Report* (1892).

The mechanism invented by Professor Peddie (*q.v.*) for the solution of an equation of the n^{th} degree depends on the principle involved in a well-known system of pulleys. The cords, instead of being fixed at one end to a rigid bar, are wound round drums attached to this bar. When amounts proportional to the coefficients of the terms in the given equation are unwound and the arm turned through an angle θ so that a spring-drum, to which the free end has been previously attached, resumes its initial position, a measurement of this angle θ will enable us to obtain a root of the equation.

In other instruments for which R. Skutsch has proposed the name "Equation-Balances," the position of equilibrium of a solid body, or of a system of solid bodies to which weights proportional to the coefficients of the given equation are attached, is sought for. In a certain number of these only one beam is employed. It is then necessary that the distances of the forces from the fixed point may be modified proportionately to the different powers of the variable. Among apparatus of this kind may be mentioned

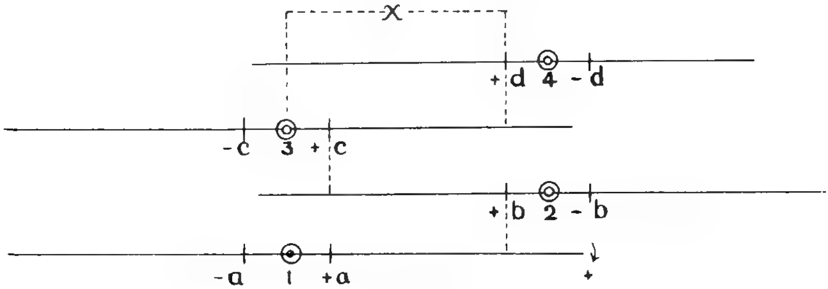


FIG. 3.

the instrument invented by C. Exner, which can solve all equations of the first seven degrees.

In the case of equation-balances which depend on the equilibrium of a system of bodies, there are as many beams as terms in the equation. Each of these beams carries a weight representing the coefficient corresponding to its numerical order in the equation. The distances of the different weights from the axes of rotation of their respective beams are always equal, and each beam rests on the preceding at a distance x from its axis of rotation. It is this variable distance x which furnishes the value of the unknown when the system is in equilibrium.

The machine invented by C. V. Boys[1] was one of this kind.

Let the levers be called successively 1, 2, 3, 4. Then 1 is on a stationary axis, and has at unit distance from this point, and on either side of it, pivots from each of which hangs a pan or hook marked $+a$ and $-a$. A second beam, 2, is connected with 1 by a sliding joint, which is permanently at unit distance from the axis of 2. Let this joint also carry a scale pan, and let there be another at unit distance on the other side. These are marked $+b$ and $-b$. If a weight b be put in either of the latter pans it will produce a turning moment on b of $\pm b$ units and on a of $\pm bx$ units, where $x+1$ is the distance between the axes of 1 and 2. Such a pair of beams will solve a simple equation

[1] *Phil. Mag.* (5) xxi. (1886), p. 241.

$a \pm bx = 0$, for, as the second beam is made to move, the sliding joint must pass some point where $a \pm bx$ is zero. The addition of another beam, 3, will enable us to solve a quadratic, a fourth a cubic, and so on.

In the case in which a quadratic has no real roots it is claimed that the machine can still be employed to find the imaginary roots. The same applies in the case of a cubic equation with only one real root; but for equations of higher degree the machine, though capable of determining the real roots, is incapable of finding the imaginary ones.

The mechanism invented by L. Torres gives not only the real roots, but also the imaginary roots of an equation. His instrument plays the same part among machines for solving equations that the logarithmo-graphic method plays among the methods serving to solve equations graphically. A logarithmic scale and a regular scale are rolled on separate drums. The two drums are then mounted on the same axis, and they are so connected that when the drum on which the logarithmic scale is wound has made one turn, that on which the regular scale is wound advances one division. The whole formed by these two drums is called by the inventor a "Logarithmic Arithmophore." The first of the two drums corresponds to the characteristic, the second to the mantissa. A first arithmophore, on which the value of x is noted, is united mechanically to the other arithmophores on which are the values of the coefficients in the equation

$$A_m x^m + A_{m-1} x^{m-1} + \ . \ . \ . \ + A_1 x + A_0 = 0.$$

Of these coefficients A_p, $A_{p'}$, $A_{p''}$, . . . are positive, and the others, A_n, $A_{n'}$, $A_{n''}$, . . . are negative. As the arithmophore of the variable x is turned, a convenient mechanical construction brings into view on two special arithmophores the values of the polynomials

$$\begin{aligned} P &= A_p x^p + A_{p'} x^{p'} + \ . \ . \ . \\ N &= A_n x^n + A_{n'} x^{n'} + \ . \ . \ . \end{aligned}$$

When the arithmophore of the variable x indicates a value for which the values of P and N are equal, this value of x is a positive root of the equation. The particular mechanical medium devised by Torres to obtain this result is a special fusee, which accomplishes for the mechanical calculation the principle of logarithmic addition, just as the curve of logarithmic addition does so for the graphical calculation.

The first model which Torres constructed in 1893 gives the solution of equations of the form $x^9 + Ax^8 = B$ or $x^9 + Ax^7 = B$. Since then he has noticed that when the equation in question is a trinomial the special fusee may be dispensed with.

Mechanisms for the Solution of Systems of Linear Equations

This same machine can be so constructed as to solve linear systems with several unknowns. Another mechanism for this purpose was invented by Lord Kelvin.[1]

This apparatus consists of n rods, each supported on a knife edge on a fixed axis. Each rod carries n pulleys, which can be adjusted by means of

[1] *Proc. Roy. Soc.*, xxviii. p. 111; Thomson and Tait's *Natural Philosophy*, 2nd ed., p. 482.

geometric scales. Over these are passed in a certain order n threads kept stretched by convenient weights. The angles turned through by the rods, as a result of the changes in length of the threads, determine the roots. It is claimed that the actual construction of such a machine would be neither difficult nor complicated. A fair approximation to the root being found by a first application of the machine, the residual errors may be easily calculated. The machine may then be applied (without changing the positions of the pulleys) to find the necessary corrections, so that there would be no limit to the accuracy thus obtainable by successive approximations.

Hydrostatic Solution of Equations or Systems of Equations

A. Demanet has indicated a method of solution of trinomial equations which depends on the use of vessels of convenient forms.

To solve an equation of the third degree of the form

$$x^3+x=c,$$

where c is a constant, an inverted cone and a cylinder, joined together by means of a tube, are taken.

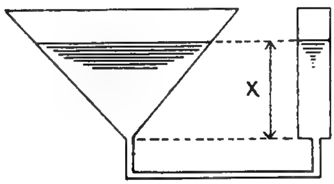


Fig. 4.

The radius R of the cone and its height H are in the ratio

$$R : H = \sqrt{3} : \sqrt{\pi},$$

while the base of the cylinder is taken as 1 sq. cm. If c cubic centimetres of water are poured into one of the two vessels, the water will rise to the same height h in both. The volume of water contained in the cone will be h^3, that in the cylinder h, so that we have

$$h^3+h=c.$$

By measuring the height h of the water we thus obtain a solution of the equation.

In the case of the equation

$$x^3-x=c$$

the cone alone is used, and a solid cylindrical piece whose base is one sq. cm. is introduced. The volume c of water poured in will thus be the difference between h^3 and h, and therefore h, the height of the liquid, is again a solution.

By a substitution $z=x\sqrt{p}$ we can reduce all reducible equations of the third degree such as $z^3+pz=q$, where p and q are given positive numbers, to the form $x^3+x=c$.

Again, by means of the hydrostatic balance devised by G. Meslin any equation of the form

$$px^m+qx^n+. \quad .=A$$

may be solved.

It consists of a beam on which are suspended solid bodies with axes vertical, whose forms and dimensions are such that the volumes immersed, when x units of length are sunk in the liquid, are proportional to x^m, x^n . . . These solid bodies are fixed at distances from the axis of rotation of the beam respectively proportional to $|p|$, $|q|$, . . . to the right or the left of this axis, according to the sign of the corresponding coefficient in the equation. Having equilibrated the balance, we next suspend at unit distance from the axis of rotation a weight equal to $|A|$. The equilibrium is disturbed, but is re-established on allowing water to enter the vessels by means of the tubes. If h is the height immersed when equilibrium is restored, the thrust on the

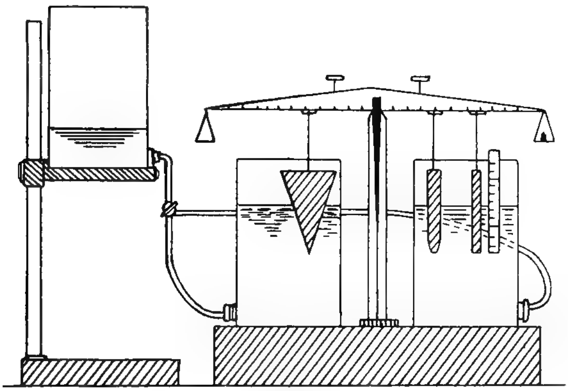


FIG. 5.

solids will be represented by h^m, h^n, . . . and their moments with respect to the axis of rotation of the beam ph^m, qh^n . . . Since there is equilibrium

$$ph^m+qh^n+ \ . \ . \ . \ =A,$$

so that h is a solution of the equation.

By adding more water the equilibrium will again be disturbed, but when a sufficient quantity has been added it will be again restored, and thus another root will be obtained.

Electrical Solution of Equations

Felix Lucas has shown that the roots, real or imaginary, of any algebraic equation with real numerical coefficients may be obtained by a single graph and without calculation by the aid of an electrical process.

Let $F(z)=0$ be the given equation of degree n. Let $\lambda_1, \lambda_2, \lambda_3, \ldots \lambda_{n+1}$ be any $n+1$ real unequal numbers, and let

$$f(z)=(z-\lambda_1)(z-\lambda_2) \ . \ . \ . \ (z-\lambda_{n+1}).$$

Decomposing $F(z)/f(z)$ into partial fractions, we get

$$\frac{F(z)}{f(z)}=\frac{\mu_1}{z-\lambda_1}+\frac{\mu_2}{z-\lambda_2}+ \ . \ . \ . \ +\frac{\mu_{n+1}}{z-\lambda_{n+1}}$$

where $\mu_1, \mu_2, \ldots \mu_{n+1}$ are all real and definite.

Now mark in the plane P of the complex variable z the points $l_1, l_2, \ldots l_{n+1}$ on the real axis, having for abscissæ $\lambda_1, \lambda_2, \ldots \lambda_{n+1}$. If, then, we charge each of the points l_i with a quantity of electricity μ_i, the nodal points of the equipotential lines traced on the plane P will be the root-points of the equation $F(z)=0$. The equipotential lines on the conducting plane may be determined by means of a galvanometer, or they may be sketched electrically by an electro-chemical method.

Lucas remarks that if an integral function of degree $n+2$ is taken for $f(z)$, the electro-magnetic method corresponding to this choice of $f(z)$ is very easy. If iron filings be scattered on a sheet of paper, the lines of force of the magnetic field can then be traced out. The root-points sought will be the points where the magnetic force is zero.

Another method is that devised by Russell and Alty.[1]

Let

$$f(x)=a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0 = 0.$$

Choose n quantities

$$b_1, b_2, \ldots b_n,$$

so that

$$b_1 + b_2 + \ldots + b_n = -a_{n-1}/a_n.$$

Then

$$\frac{f(x)}{(x-b_1)(x-b_2)\ldots(x-b_n)} = a_n + \frac{A_1}{x-b_1} + \frac{A_2}{x-b_2} + \ldots + \frac{A_n}{x-b_n},$$

where

$$A_1 + A_2 + \ldots + A_n = 0.$$

Consider the magnetic field round a long vertical wire carrying a current of C amperes, and suppose the earth's horizontal field in the neighbourhood is uniform and that its horizontal intensity in C.G.S. units is H. The magnetic force at any point P at a perpendicular distance of r cms. from the axis of the wire will be the resultant of a force $C/5r$ acting at right angles to the plane containing r, and the axis of the wire and a force H directed to the magnetic pole. There is always a neutral point on the line through the axis of the wire perpendicular to the magnetic meridian. If x be the distance of this point from the axis, then

$$x = C/5H.$$

Now suppose n wires are arranged in a plane perpendicular to the magnetic meridian, and let them cut another plane perpendicularly at points whose distances from a fixed point are $b_1, b_2, \ldots b_n$. Then if $C_1, C_2, \ldots C_n$ be the values in amperes of the currents in these wires, and X, Y the components of the resultant magnetic force at (x_1, y_1), then

$$-X = \frac{C_1}{5r_1} \cdot \frac{y_1}{r_1} + \frac{C_2}{5r_2} \cdot \frac{y_2}{r_2} + \ldots$$

$$Y = H + \frac{C_1}{5r_1} \cdot \frac{x_1 - b_1}{r_1} + \frac{C_2}{5r_2} \cdot \frac{x_1 - b_2}{r_2} + \ldots$$

where

$$r_m{}^2 = (x_1 - b_m)^2 + y_1{}^2$$

[1] *Phil. Mag.*, 6th series, xviii. (1909)

Hence

$$Y+iX=H+\frac{C_1/5}{x_1+iy_1-b_1}+\frac{C_2/5}{x_1+iy_1-b_2}+\ \ldots$$

At a neutral point $X=Y=0$. Hence if (x_1, y_1) is a neutral point, then x_1+iy_1 is a root of the equation

$$0=H+\frac{C_1/5}{x-b_1}+\frac{C_2/5}{x-b_2}+\ \ldots$$

Hence if $C_1, C_2, \ldots$ be so adjusted that $C_n=5HA_n/a_n$, then x_1+iy_1 will be a root of $f(x)=0$.

An exceedingly ingenious method, the invention of Arthur Wright, M.I.E.E., is described in the same volume (p. 291). This device depends on the use of slide resistances. The principle of the ordinary logarithmic slide rule is combined with addition and subtraction, by utilising the laws according to which resistances combine in series or parallel. The products found by the slide-rule method are represented either by the resistances or by the reciprocals of the resistances of certain wires. As an adequate account of this instrument could not be given within the scope of this article, the reader is referred to the original memoir. There it is shown how to solve cubic equations, equations of higher degree than the third, equations containing miscellaneous functions, and transcendental equations; also how to trace any curve electrically. It is claimed that this machine can evaluate almost all mathematical expressions; and, the writers add, it seems particularly suited to harmonic analysis, as the integrals representing the coefficients of sin nx and cos nx in the expansion of $f(x)$ can be readily found.

[Further information and references to original works may be found in an article on this subject in the *Encyclopédie des Sciences Mathématiques, Pures et Appliquées*, Tome I. vol. iv. Fasc. 3.]

(1) Apparatus for Solving Algebraic Polynomial Equations.

By R. F. Muirhead, D.Sc.

The geometrical principle on which this is based is illustrated by the diagram in the case of three simultaneous equations in three unknowns.

To explain it, let the three equations to be solved be:

$$a_1x+b_1y+c_1z+d_1=0,$$
$$a_2x+b_2y+c_2z+d_2=0,$$
$$a_3x+b_3y+c_3z+d_3=0.$$

Here $X_1'X_1Y_1'Y_1Z_1'Z_1O_1$ is a straight line, and X_1X, $X_1'X'$, etc., are straight lines perpendicular to it.

We have $X_1'X_1=Y_1'Y_1=Z_1'Z_1=1$, and $X_1O_1=x$, $Y_1O_1=y$, $Z_1O_1=z$.

On $X_1'X'$ we lay off $X_1'A_1=a_1$ and draw A_1X_1 to meet $Y_1'Y'$ in P_1, Y_1Y in y_1, and O_1O in F_1.

On P_1Y' we lay off $P_1B_1=b_1$ and draw B_1Y_1 to meet $Z_1'Z'$ in Q_1, Z_1Z in z_1, and O_1O in G_1.

On Q_1Z' we lay off $Q_1C_1=c_1$ and draw C_1z_1 to meet O_1O in R_1.
On R_1O we lay off $R_1D_1=d_1$.
Then $O_1F_1=a_1x$, $F_1G_1=b_1y$, $G_1R_1=c_1z$, and $R_1D_1=d$,

$$\therefore \quad O_1D_1=ax_1+by_1+cz+d_1.$$

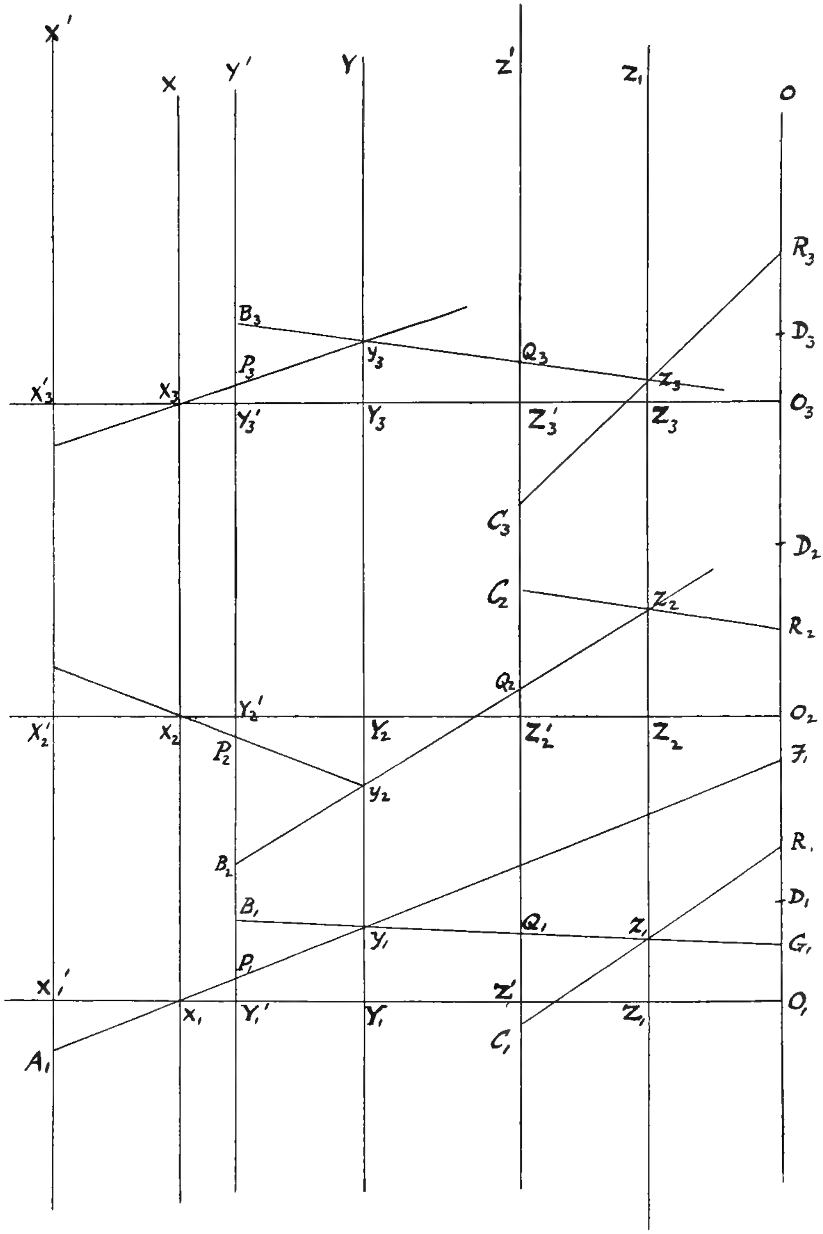


Fig. 6.

The figure indicates also similar constructions for

$$a_2x+b_2y+c_2z+d_2 \quad \text{and} \quad a_3x+b_3y+c_3z+d_3,$$

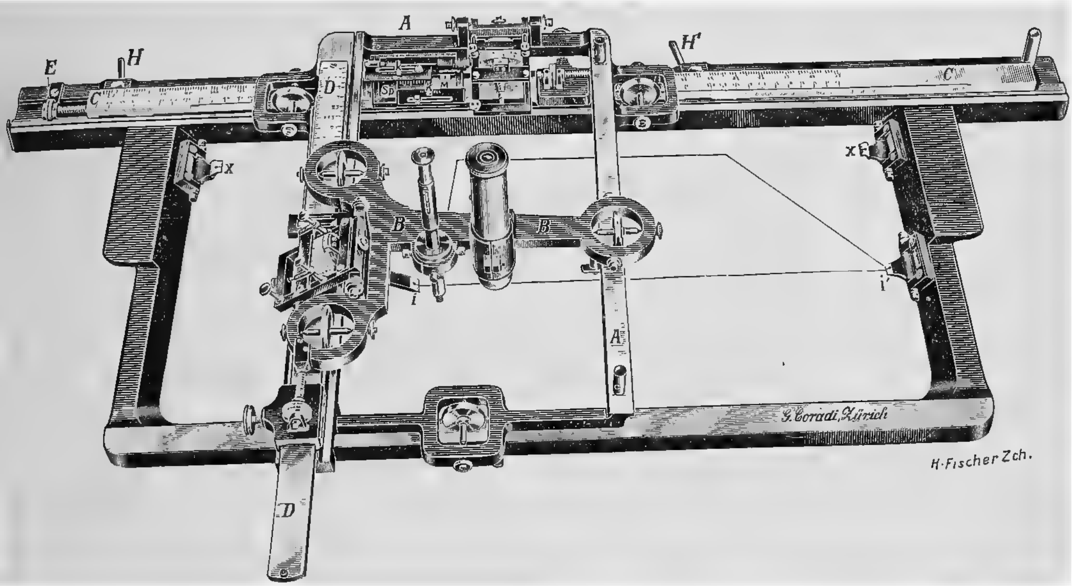
showing that these are represented by O_2D_2 and O_3D_3 respectively.

If now x, y, and z be successively or simultaneously varied so that D_1, D_2, D_3 coincide with O_1, O_2, O_3 respectively, the values which x, y, z then have will be the solution of the given equations.

(2) Equation Solver. Exhibited by Professor W. PEDDIE.

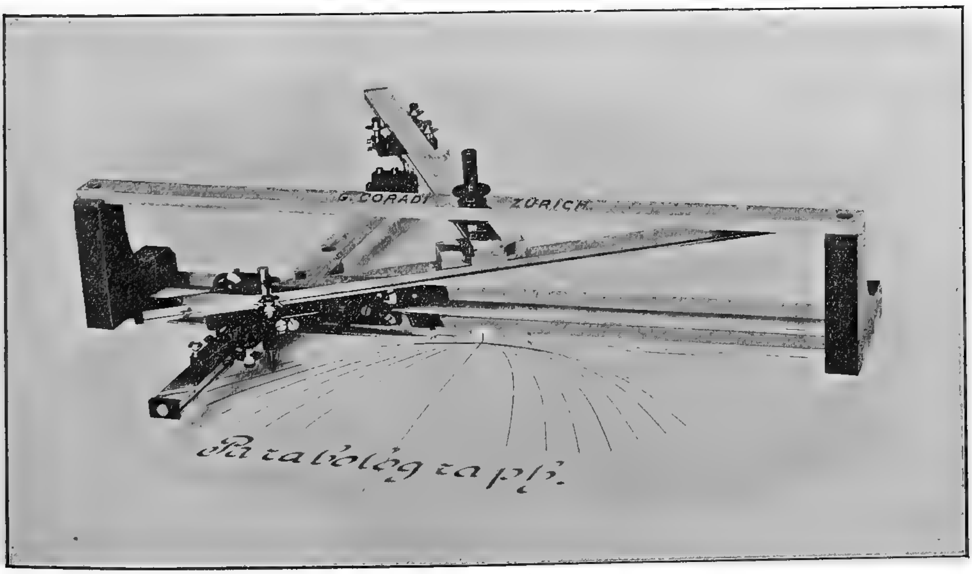
XI. Instruments for Plotting.

(1) Co-ordinatograph. Exhibited by G. CORADI, Zürich.



INSTRUMENT for plotting points by their rectangular co-ordinates. May also be employed for the construction of curves with given ordinates.

(2) The Payne-Coradi Parabolograph. Exhibited by G. CORADI, Zürich.



(Designed by Professor Henry Payne in Melbourne.)

(3) A Roller Protractor. By A. Ott, Kempten, Bavaria.

This protractor consists essentially of a graduated rule, which may be rotated about one extremity on the drawing board, and further, of a measuring roller running on the paper. This is illustrated in fig. 1, where the instrument is shown when put together. The separate parts are shown in fig. 2. These are the rule L and the roller frame R, which are coupled together only when the instrument is in use. It then turns about a centre formed by the pointed pin *p*. This pole is provided with the weight *g*, to secure the position of the instrument during operations.

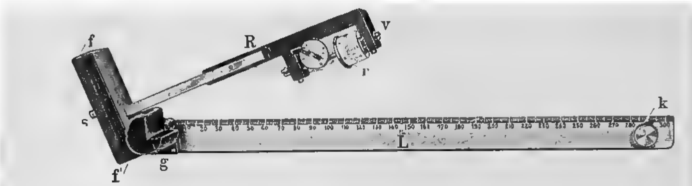


Fig. 1 —Protractor for Polar Co-ordinates.

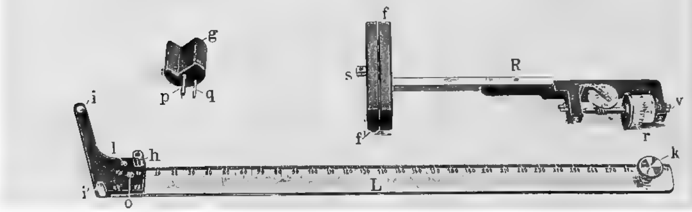


Fig. 2.—Single Parts of Protractor.

The rule L is 12 inches long, and bears on the bevelled edge a suitable graduation. It is fixed to the drawing by setting the pin *p*, which marks the centre, in the socket *h*. The roller frame R is then connected with the rule by placing the small ball pins *f* and *f′* in the sockets *i* and *i′*. In moving the rule round the pole, the measuring roller R makes twelve rotations for one full turn of L, or one rotation for an angle of 30 degrees. The vernier permits the angles to be read to single minutes, while a small graduated disc counts the number of complete rotations of the roller.

Adjustment and Use of the Protractor

After having put together the instrument and having set the reading to zero, draw a fine line along the bevelled edge of the rule. Rotate the rule carefully by the knob *k* through an exact revolution back to its original position. If the adjustment of the instrument is correct, the final reading of the roller must again be zero. If it is over that, say by ten minutes, the distance of the rim of the roller from the pole must be diminished by screwing back the adjusting screw *s* by about one-fifth of a turn. Should the reading be below zero, the screw *s* must be turned forward. By repeating this operation the protractor may, in a very short time, be so adjusted that after ten complete revolutions of the rule the reading will hardly be one minute out—an accuracy that fully answers all practical requirements. The instrument is then ready for use.

XII. Pantographs.

(1) Precision-Pantographs. By A. Ott, Kempten, Bavaria.

The Precision-Pantograph can be used for enlarging and reducing drawings in all ratios between 20 : 1 and 5 : 4, and, with the more perfect instruments, from 20 : 1 to 2 : 3. It can further be so set as to compensate

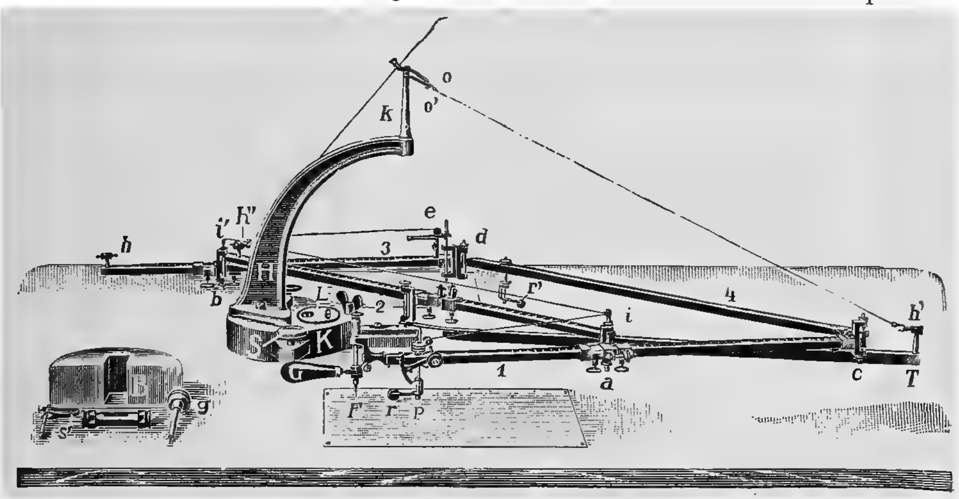


Fig. 1.—Precision Pantograph.

for any shrinkage of the paper of old drawings, so often met with. The pantograph consists of a heavy crane-shaped iron standard consisting of a bow H, a weight B, and a sole plate K. From the top of the standard are suspended, by a couple of thin wires, four bars, 1, 2, 3, 4, of hard-drawn brass tube, connected with each other by pivot-joints and partly supported by a fifth bar T.

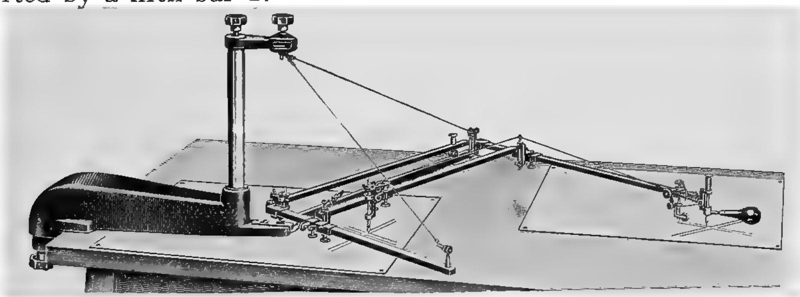


Fig. 2.—Precision Pantograph with Projecting Standard.

The axis of rotation *pk* is set vertical by the levelling screws S and the levels L.

The four bars form a parallelogram which, at one corner, moves round a ball-joint, as illustrated in fig. 1. Two of the pivot-joints are mounted on sleeves that can slide along the bars, while one joint bears the so-called pole ball. The sleeves are provided with verniers and micrometer adjust-

ments for accurate setting to the respective ratios. The bars 1, 2, and 3 bear a millimetre scale and a number of index marks for the setting of various ratios. The bars 1 and 2 are further provided with the necessary guides for the tracing pin and the pencil, the latter guide being mounted on a movable sleeve similar to those on bars 1 and 3. The instrument may be mounted either with the pole at the end or in the centre.

(2) Pantograph. By CAREY, London.

Exhibited by the MATHEMATICAL LABORATORY, University of Edinburgh.

XIII. Watkins' Instruments for Calculating Times for Photographic Exposure and Development.

THESE all use logarithmic scales arranged as described below :—

The Standard Exposure Meter (the earliest pattern, invented in 1890) has four logarithmic scales, one for each of the factors, viz., plate, diaphragm, actinometer, and exposure. A separate pointer (one for each factor) indicates

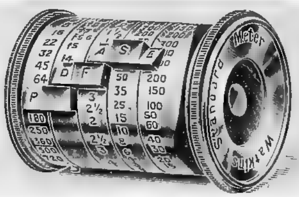


FIG. 1.

on each scale, the two end scales being fixed on the body of the instrument, while the two central scales revolve with the movable pointers. After the separate pointers P, D, and A are set to the required values, the final pointer E indicates the cumulative result on the final exposure scale.

The Watch-shaped Bee Meter does without pointers, but has logarithmic scales for the same four factors. The centre disc is revolved until the stop

FIG. 2.

(or diaphragm) value is set against the plate (speed) value. The final exposure result is read against the light (actinometer) value.

The Factorial Calculator is used for calculating development by the Watkins' factorial method. The two circular scales are divided logarithmically into sixty parts. The pointer being set to the multiplying factor (9 in the illustration), the required time of development (usually in minutes) is read

on the outer scale against the figure on the inner scale which represents the time of appearance of the image—usually seconds. The division of the circle into sixty parts automatically translates seconds into minutes in the result.

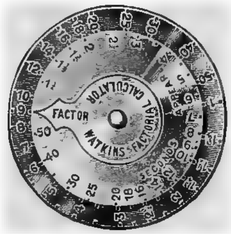


FIG. 3.

The Time Thermometer (fig. 4) utilises a logarithmic scale in an interesting way. It has been found that the correct times of development are indicated against an even division temperature scale by a logarithmic time scale, as shown in fig. 5. Two things require to be settled to make

FIG. 4.

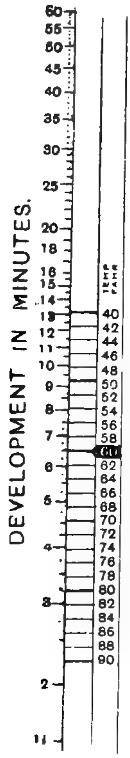


FIG. 5.

this scale. Firstly, the right time of development for one given temperature ($6\frac{1}{2}$ minutes for 60° F. in this diagram); and secondly, the temperature coefficient of the developer, that is, the time ratio for the same result at two temperatures 10° C. apart. In this case the temperature coefficient is 1·9. In the time thermometer illustrated, the temperature scale is omitted and

18

a logarithmic scale of times (minutes' development) is placed alongside the column of mercury, so that the requisite time for development is read off without any calculation when the thermometer is dipped in the developer.

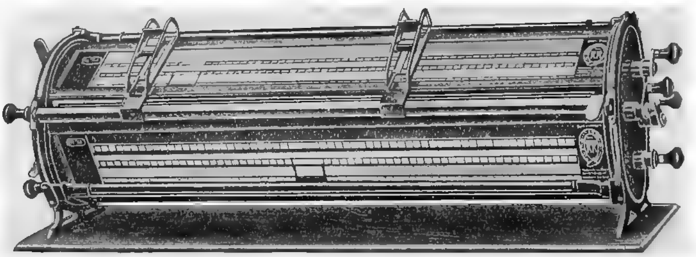
XV. Miscellaneous Group.

(1) The Robertson Rapid Calculating Machine Co., Ltd.

THIS calculator is a type of ready reckoner, and is manufactured in Glasgow. The actual machines, so far as they are already produced, are not yet on the market for general sale, but have been designed for the company's own use.

A ready reckoner is helpful in a certain way, but the idea contemplated was to go entirely beyond the scope of it, and at the same time to teach arithmetic by the use of equivalents in all sorts of measures, and to train the operator by educating his eye.

This machine itself is a mechanical device for displaying printed tabulated matter, and is capable of showing an almost unlimited number of totals within a reasonable compass.



The New "RR" Machine.

The present model, as illustrated, is set upon a desk-table. It has four distinct faces, each face showing different sets of equivalents. The operator, by simply pressing a small key, brings the required face opposite him, with the controlling handles ready for use.

Each face of the machine with its printed records may be likened to a book with 200 or 300 pages open at the one time, allowing the machine to be operated, while showing the full sets of equivalents. The operator is thus enabled in many instances to do some thirty different calculations in five minutes, without requiring to re-set the machine.

To the sloping desk in front of the machine is fitted a further series of calculated records of equivalents, in order to enable the operator, having found an answer in the main machine, to convert it into other equivalent values.

(2) A Direct-Reading Instrument for Submarine-Cable and other Calculations. By Rollo Appleyard, M.Inst.C.E.

In predetermining the speed of signalling through submarine cables, and the relationship between that speed and the cost of the conductor and of the dielectric of the cable, the principal term is log D/d, where D is the diameter of the dielectric, and d is the effective diameter of the conductor. This term also appears when calculating the capacity constants and the dielectric-resistance constants of a given dielectric, from tests of the cable-core; and it enters into problems relating to the transmission of electrical energy through cables.

The instrument here described depends upon the use of a logarithmic spiral, the pole of which is at the centre of a circle. This circle is divided into degrees, and the two radial arms, each of which is free to turn about the centre independently of the other, can thus be set to any required angle. Each radial arm is provided with a scale of equal divisions, and the zero marks of these scales are always at the pole of the spiral, *i.e.* at the pivot of the arms. One radial arm can be allotted to D, and the other to d, and they can be set to intersect the spiral to correspond with a given pair of values of D and d, as read upon their respective scales. If the angle between the radial arms be denoted by $(\theta_1-\theta_2)$, for direct-reading the shape of the spiral must be such that for all pairs of values of D and d, this angle $(\theta_1-\theta_2)$ must be proportional to log D/d.

The constants of the spiral are discussed, and a method is explained for magnifying the spiral in the neighbourhood of the pole, so as to get accurate readings. See *Proc. Inst. Civil Engineers*, vol. clxxxiv. part (ii.); also *Proc. Physical Soc.*, vol. xxiv. part ii.

(3) A Pocket Calculator, "Espero," 6 inches × 3 inches. Lent by Andrew Wilson, M.Inst.C.E.

This is really a form of abacus. The instructions are printed in Esperanto. It was sold at Cracow in 1912 at the Congress which celebrated the jubilee of Esperanto.

The method of using it is as follows:—In order to add two numbers they are pulled down by the spike supplied. When the column shows black, move the spike to the top of the scale and advance the next column by unity. The sum is read at the foot of the columns.

(4) The Napierian "Bones" rendered "All Mechanical." By George Thomson.

The "Bones" were invented by Napier for performing multiplication. In doing this, it was necessary to add mentally the "units" on one stick to the "tens" on another stick.

Thus, suppose the sticks Nos. 3, 4, and 7 to be put together to form the

multiplicand 347, with an index rod containing the multipliers placed underneath, as shown below :—

			1	1	1	2	2	2
3	6	9	2	5	8	1	4	7
		1	1	2	2	2	3	3
4	8	2	6	0	4	8	2	6
	1	2	2	3	4	4	5	6
7	4	1	8	5	2	9	6	3
	2	**3**	**4**	**5**	**6**	**7**	**8**	**9**

Then, by this arrangement, the first digit (right hand) of, say, 3 times 347 is found above the multiplier 3. The second is found by adding mentally the two figures at the junction above, viz. 2 and 2, which gives 4. The third is found by adding mentally the two figures at the next junction, viz. 1 and 9, which gives 10. Thus, above the multiplier 3, we can read off 1, then 4, then 10, which is written from right to left as 1041.

Sometimes the two figures at a junction amount to more than 9, as, for instance, in column 7, where 8 and $4=12$. This is read " 2," while carrying the " 1 " to the next junction, which means the mental addition of *three* figures at the said junction, viz. $2+1+1=4$.

In the original " Bones " of Napier, and in modifications of the same, the mental work here spoken of is necessary ; and it is the purpose of the improvement here exhibited to dispense with it.

The cards are arranged to form a multiplicand, as explained above, and an index card of multipliers is placed at the foot, while the broad title-card is placed at the head.

To read off, say, six times the given number (to which the cards are set) proceed thus :—

Immediately above the multiplier will be found the first digit (right hand) of the answer. This figure is enclosed in a triangular space.

In passing out of this space by the " gate " into the triangular enclosure above, the *second* digit will be found in the " gateway." This " gate " leads into another triangular enclosure above, at the outlet " gate " of which will be found the *third* digit of the answer : and so on, until the last digit is found on the title-card at the top.

In this way the mental additions necessary in the case of the older arrangements are dispensed with.

(5) A Surface Measuring Tape Line. By George Thomson.

This tape line is so graduated as to give the half square of any line measured by it, and is for finding the area of any rectangle, *without multiplication*, in the following manner :—

Measuring along the side A of a rectangle, and continuing the measurement along the side B, gives the half square of A+B; and measuring the diagonal of the rectangle gives the half square of that diagonal.

Subtracting the half square of the diagonal from the half square of A+B gives AB, the area of the rectangle sought.

Thus,

$$\frac{(A+B)^2}{2}-\frac{A^2+B^2}{2}=AB.$$

The other side of the tape is graduated in feet and inches.

Section H

RULED PAPERS AND NOMOGRAMS

I. Ruled Papers. By E. M. Horsburgh, M.A.

Ruled papers may be obtained in many forms—squared, rectangular, logarithmic, semi-logarithmic, triangular, degree-polar, and radian-polar papers are all available, and all useful.

Of these squared paper is probably the best known. It is frequently ruled in inches and tenths, or centimetres and fifths. It is used in every school, and is familiar to everybody. The recent popularising of squared paper in this country has been due largely to the writings of Professor Perry. The following is a quotation from his *Calculus for Engineers*. It comments on a difficult but clever article which he had been reading.

"The reasoning was very difficult to follow. On taking the author's figures, however, and plotting them on squared paper, every result which he had laboured so much to bring out was plain upon the curves, so that a boy could understand them. Possibly this is the reason why some writers do not publish curves. If they did there would be little need for writing."

Squared paper is extremely useful in teaching beginners the rudiments of co-ordinates, including loci and trigonometry, and, in particular, graphs. Its importance at this stage can hardly be overestimated. The pupil may be made to feel that he has embarked on a voyage of discovery, and the stimulating effect of this is considerable.

If, however, this "plotting by points" is carried on in teaching mathematics to more advanced classes, its effect may be bad, as it may lead to purely mechanical work, which does not develop the reasoning powers. At the same time, squared paper has its uses in the teaching of pure mathematics to higher classes in school. It is hardly necessary to refer to the training which may be given by the use of different scales on the axes of reference. The first ideas of "limits" may be introduced by its means. The values of a function may be plotted in the neighbourhood of a limit value, say, at $x=a$, as x gradually approaches a, and the idea of the limit is suggested. The tangent and the gradient may be made clear by simple calculations on squared paper. Areas are easily measured, and this suggests integration. Thus the way is paved for the calculus. Its uses in applied mathematics are referred to later.

One might perhaps at this stage draw attention to the rather ambiguous way in which the word "graphing" is used. It is employed to denote

(1) the sketching, not to scale, of the graph of a function from mathematical first principles, as contrasted with (2) the plotting laboriously of a number of points, and assuming an arc of a curve through these points as the graph of the function, when the former method was all that was required, and (3) the construction of a diagram to solve some practical problem, or to illustrate the results of some experiments.

The first of these three headings might be called Graphing, the second Plotting by Points, and the third Graphic Methods. As regards elementary teaching under the first of these headings, a passing reference might be made to the chapters on Graphs in Chrystal's *Introduction to Algebra*, and Functions of Real Variables in Hardy's *Pure Mathematics*. The beginner soon becomes familiar, through the graph, with important properties and peculiarities of the function, just as he recognises an individual whose "graphic" appearance presents some peculiarity. Few elementary branches of mathematics may be made more interesting than this. Ruled papers should not be used in graphing, which is essentially the determination of the *general form* of the curve.

Graphic Methods are peculiarly the province of the engineer and the experimenter. The aim is to construct a diagram from which measurements may be made and useful results deduced. Nomography is a branch of graphic methods.

Large sheets of paper, useful for computing purposes, are obtainable. These are ruled homogeneously and very faintly in small rectangles, each just large enough for two digits of the size usually written in calculating. This is a considerable help in arithmetic, as it conduces to neatness and method, important factors in work of this sort.

The use of squared paper may simplify the work of engineering drawing by avoiding the use of T- and set-squares. Various kinds of section paper, ruled in eighths and twelfths, are used in this country for mechanical design. Such papers should not be used for graphic methods, as they lose all the simplicity of the decimal system, owing to the difficulty of interpolation.

In graphic methods some further cautions which might be mentioned are the following. Too many values marked on the axes of reference are distracting to the eye, and those shown should be, as far as possible, multiples or submultiples of ten. A mistake of which it is difficult to break the beginner is drawing on too small a scale and so sacrificing accuracy. The most important caution of all is to see that no time is wasted on any unnecessary work. When the student has learnt this he is no longer a beginner. Badly chosen scales, diagrams either ridiculously small, or ill-proportioned, or else "run off" the paper, bad graduation, bad drawing, and, above all, unnecessary calculations, are a few of the ways in which time is wasted. Needle points should be used for plotting, and a small circumscribed circle should indicate the position of the point. It may seem trivial to refer to such things as blunt points and stumps of pencil, but success is only attained by attention to details. The standard of graphical work should be such as would meet with approval in a civil engineer's office.

There are many uses for decimally divided squared paper. The most obvious is the plotting of tables of statistics, or recording the results of a series

of experiments. This shows at a glance how the experiments agree with one another. As a general rule the plotting determines a definite curve, which shows how the function represented varies with respect to its argument.

It may happen that one or two of the points plotted are far removed from the curve drawn through the remaining points. This suggests that some form of error has probably occurred, and indicates the advisability of repetition.

It is rarely possible to grasp at once the full significance of a table of statistics, but when it is plotted as a graph the salient features are apparent instantly. The curve through a number of points is usually best put in by hand, the drawing being done from the concave side of the curve. The power to shift and turn the loose sheet of ruled paper is an advantage over the fixed drawing-board with its T- and set-squares.

Many observers join up the points plotted by portions of straight lines, but this is not a satisfactory method; while, on the other hand, the indiscriminate use of the "smoothing iron" must be guarded against. It might be urged that there is little mathematical justification for the use of the "smoothing iron," even though its application is almost universal. Suppose, for example, that half a dozen observations have been made in order to plot a function which happens to be represented by a first and a thirteenth harmonic of approximately equal amplitudes. If a smoothed curve were drawn through the plotted points it would probably bear a very slight resemblance to the true shape of the curve.

A few additional examples of the applications of squared paper may be mentioned.

1. *Approximations to the Roots of Equations.*—In approximating to the root of an equation, transcendental or algebraic, an approximation to three-figure accuracy is usually obtained easily and rapidly by squared paper. Thus the characteristic of $F(x)=0$ may be broken up as in $F(x)\equiv f(x)-\phi(x)$, and the graphs of $y_1=f(x)$ and $y_2=\phi(x)$ plotted near the points of intersection. The abscissæ of these give the approximations required, and the root may thereafter be delimited to any required accuracy by the "chord and tangent" method, or any other well-known rule.

2. *Tabulation by Graphical Interpolation.*—Take as an example the logarithmic function, and suppose that an elementary method is required. By means of a table of square roots the values of $10^{\frac{1}{2}}$, $10^{\frac{1}{4}}$ . . . , $10^{\frac{1}{32}}$ may be written down, and hence by multiplication $10^{\frac{3}{32}}$, $10^{\frac{5}{32}}$, etc. Turning the indices into decimals and plotting these against the corresponding numbers, a set of points on the logarithmic graph is obtained. By careful drawing four-figure logarithms may be read off or tabulated.

As regards the tabulation of functions in general, a few values of any required function may be calculated, and plotted on a large scale. If a smooth curve be drawn through these points, a table may be read off with ease and rapidity to nearly four-figure accuracy.

3. *Areas.*—Areas and traverses may be set out rapidly on squared paper. Rectilinear areas are easily reduced to the equivalent triangle, and so determined. Areas with curvilinear boundaries may either be planimetered,

or such methods as the trapezoidal or Simpson's rules may be adopted, or even the elementary one of counting squares.

If the large squares be selected judiciously, this last method need not be trivial and laborious, though it may easily be made so.

An admirable training for the eye is given by the method of "equalising up" the curved boundaries of an area, *i.e.* by replacing the curved boundary by equalising straight lines, and then treating the area as a polygon. Another example of the same training is the rapid determination of the generalised arithmetic mean ordinate over some range of the argument by means of a fine stretched thread.

4. *Empirical Formulæ.*—A useful application is the determination of a suitable function to represent empirically a given table of values. Such a function may be written down by inspection, say by Lagrange's Interpolation Formula. The result, however, is so clumsy as to be valueless in most cases, since a function of simple form is desired to indicate the law. If the graph representing the tables be sketched roughly, its appearance should suggest some simple function containing two or three arbitrary constants. These should be as few as possible, and their use is to fit a curve of the family to the most suitable position among the points, since the observed values are not accurate, but are all affected with error. In general such corrections would require the method of least squares.

An important case occurs when there are only two arbitrary constants. The correction of errors may then be made to depend on the equation of the straight line, and practically on the stretching of a fine thread among the points in a diagram. Thus in mechanical engineering the simple straight line law $y=mx+c$ is followed in many cases, particularly in dealing with the friction of machines. By stretching a fine thread among the plotted values the most suitable values for m and c may be read, which gives the "law" of the particular machine. As another example, the group of expansion and compression curves from steam, oil, gas, and air engine indicator diagrams may be represented by $y=ax^b$ where a and b are arbitrary constants, x and y piston displacement and pressure respectively. On taking logarithms we have $\log y=b\log x+\log a$. Putting $Y=\log y$ and $X=\log x$, the points (X, Y) lie upon the straight line $Y=bX+\log a$. By using the stretched thread to determine the straight line which lies most evenly among these points, we obtain b and $\log a$, and hence a. Instead of plotting the logarithms, logarithmic paper might have been used.

Further examples of the kind where the methods of the straight-line law are useful might be indicated by $y=ax+bx^2$, $y=ax/(x+b)$, $y=ax^r+bx^s$, $y=a/(x+r)+b/(x+r)^2$, where a and b are arbitrary constants and r and s are supposed to be known. The exponential curve is an important case which is reducible to the straight-line law.

The corresponding problem involving three variables x, y, and z may be made frequently to depend on the equation of the plane. In this case a water surface may take the place of the stretched thread.

5. *Miscellaneous Uses.*—All drawing is simplified which would necessitate otherwise the use of the T- and set-square. The operations of graphical arithmetic and graphic differentiation are shortened. Definite integrals are evalu-

able by finding the area represented and then interpreting the unit square. Diagrams illustrating functions of two independent variables may be made by various forms of contour representation. Graphic statics, bending moments, and the curvature of beams and columns are three further subjects out of many in which squared paper is useful.

If it is necessary to differentiate or integrate a function which is only represented by its graph, the methods of graphic differentiation or integration must be employed. The usual treatment may be found in any text-book. If the function were known, or if its tabulated values were given, analytical or arithmetical methods would be employed. In many cases in practice, however, treatment within the limits of graphic accuracy is all that is required.

Graphic integration gives accurate results, judging by the standard of graphic work. Graphic differentiation does not. The reason for this is that an area may be determined closely, while it is difficult to draw accurately a tangent to a curve. The former is the foundation of graphic integration, the latter of graphic differentiation. In attempting to draw a tangent to a graph at the point P, the straight edge may be placed so as to form the chord of a small arc at P, where P is considered as the vertex of the arc of the osculating circle at this point. A parallel through P to the chord gives the tangent required.

An excellent test for the accuracy of one's drawing is to set out any parabola $y=a+bx+cx^2$ and differentiate this graphically. If the points found lie very closely on a straight line in a large-scale diagram, the work is satisfactory. Excellent exercises in scales and in the use of polar distances are given by this, as by many other branches of graphic methods. It must not be forgotten that the drawing of the new graph is only part of the work; it is equally important that it should be read correctly.

Logarithmic Papers

Logarithmic Papers are formed by spacing the ruled lines not equally apart, but at distances representing the logarithms of the corresponding numbers. In *semi-logarithmic* paper only one of the systems of lines is so treated. These papers are specially useful in the cases $y=ax^b$ and $y=ab^x$. A great number of practical applications may be brought under these two equations.

Triangular Papers

Triangular Papers are of considerable interest. In the most usual form an equilateral triangle is taken as triangle of reference, as in trilinear co-ordinates, and parallels, spaced equidistantly, are drawn to the three sides. This paper is useful in showing the graphical representation of three variable quantities whose sum is constant, as, for example, the degree of concentration of a mixture of three substances. Important practical applications arise in the metallurgy of ternary alloys. It is also useful as a graphic method for harmonic analysis.

Polar Papers

Polar Papers are also important. Points are plotted on these by their polar co-ordinates r and θ. Usually the circle is graduated in degrees, forming a degree-polar paper, and serving incidentally as a useful protractor.

This kind of paper has never been as popular as squared paper, probably because an area described upon it is not interpreted directly. This disadvantage is overcome by the radian-polar paper, in which the angles are set out in radians so as to simplify calculations.

Transparent Ruled Papers

These are frequently desirable, especially when it is necessary to superpose one diagram upon another. Thus they are useful in some simple methods of graphical harmonic analysis, and in testing the results of periodogram analysis, and in all cases where the shapes of two or more diagrams are to be compared.

(1) **Logarithmic Graph-Papers and their Uses.** Messrs Schleicher & Schüll (Düren). Translated by W. Jardine, M.A.

In technical and scientific literature references are made here and there to logarithmically divided paper and to its use in isolated problems and investigations, but the remark is always added that such papers are not manufactured in quantity. It is true that till recently logarithmic paper was only produced and put on the market in small quantities, and was under these circumstances difficult to obtain, as exact knowledge of where to get it was lacking. Lately, however, the well-known firm of Carl Schleicher & Schüll of Düren (Rheinland) has taken up the manufacture of logarithmically divided papers and placed them on the market. By doing this the firm has supplied a decided want, since these graph-papers are of great help in a whole series of technical and scientific problems, especially in the tracing of graphical representations (diagrams, illustrating figures, tables of isopleths, etc.), in applications of the graphic calculus, in the so-called science of nomography, which is essentially the theory of the graphical solution of numerical equations, and in the gathering together of methods for the construction of tables (Abacus).

Logarithm paper may be used

(1) in astronomical and meteorological work of all kinds,
(2) in mathematical and scientific instruction,
(3) in physical and technical practice,
(4) in calculations and graphical representations in aeronautics, *e.g.* in determining the lifting capacity and motion of a free balloon, etc.,
(5) in the tracing of discharge diagrams,
(6) in the representation of movements of capital under the influence of compound interest,
(7) in various economic, statistical, and insurance calculations,
(8) in graphical representations of statistics of population,
(9) in graphical representations of formulæ for determining mean velocities in natural water-courses,
(10) in electro-technical, photometric, etc., work.

The use of logarithm paper facilitates the work in many of these cases, and we may say that, by using these papers, many examples and relations appear in quite a new light, and that thereby are often discovered methods of representation and solution of problems which in compactness and elegance leave nothing to be desired.

We must not forget to refer briefly here to the mathematical principles lying at the base of the practical applications of logarithmically divided paper, and by a few examples to show how these applications take shape.

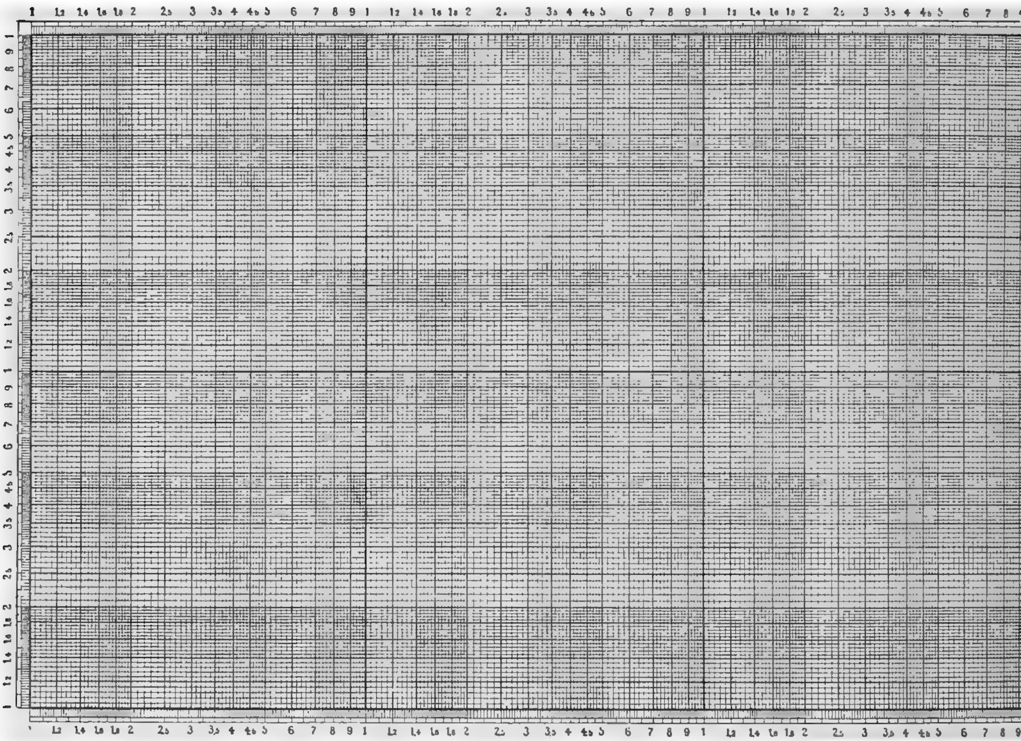


FIG. 1.

The firm of Schleicher and Schüll makes two kinds—logarithm papers which are divided linearly in the direction of the abscissæ (as in common millimetre graph-paper) and logarithmically in the other direction; logarithm papers which are logarithmically divided in both directions (see the reduced diagrams 2 and 1).

The first kind is in demand when it is required to represent and investigate some phenomenon or the change in a quantity or magnitude dependent on some other quantity, provided it is known that an exponential law lies at the base of the change, or provided at least that the change approximately follows such a law. This is frequently the case when the curve which represents geometrically the law of dependence shows no maxima or minima and no contraflexure.

Here, preferably, we have quantities which in equal intervals of the argument increase by a constant percentage, *e.g.* a sum of money laid out at compound interest; or, otherwise expressed, we are dealing with magnitudes which change in geometric progression when and so long as another magnitude, on which the first depends, changes in arithmetic progression.

All these properties of the exponential law

$$y=ae^{kx} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

(x the independent, y the dependent variable, a and k constants) are expressed in the simplest way by the corresponding form

$$\frac{dy}{dx}=ky \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

k is therefore the factor of proportionality, *i.e.* the percentage increase or decrease, and a is the initial value of y for $x=0$.

Every exponential law of the form of equation (1) will be represented on logarithm paper of the first kind (see above) by a straight line, if x is measured on the millimetre axis and y on the logarithmic axis. This is graduated from 1 to 10 or in larger sheets from 1 to 100, 1000, and so on, and is also numbered for the fractions 0·1, 0·01, etc.

That equation (1) appears as a straight line on logarithm paper follows immediately on taking logarithms

$$\log_e y=\log_e a+kx \quad . \quad . \quad . \quad . \quad . \quad (3)$$

or

$$y'=\alpha+kx,$$

where $\log_e y=y'$, $\log_e a=\alpha$. Thus we get on logarithm paper a straight line which has the gradient k, and cuts the ordinate axis $x=0$ at the point marked α.

Here we have a great number of possible applications in insurance, commercial, and statistical investigations, and in particular in practical railway construction.

Suppose we wish to find the possible profit accruing in the future from a new railway or a station site, and that we have at hand statistics of the population of some district or of the traffic returns of some goods station. If we use common millimetre graph-paper, the plotting of these numbers with the single years as abscissæ will give nothing more than a curve which mounts more or less steeply. How steep it is, and whether the increase is a definite (or, at least, for a considerable period of time, constant) percentage, can only be determined by detailed calculation, which could, moreover, be carried out without graphical representation. On logarithm paper the curve will as a rule be a straight line, that is, we can draw among the plotted points a straight line, which lies as evenly among them as possible. If this is the case, then the increase of trade is in geometric progression, just as with capital laid out at compound interest. When we determine the gradient of this approximate straight line from the graph, we get at once the percentage increase. If we cannot draw such an approximate straight line, it may be possible to trace a broken straight line through the points, and we can then assert that the trade has increased by p per cent. in the first a years, and by q per cent. in the last b years.

If we wish to make calculations from this approximate straight line we have traced, we must pay attention to the scale of the logarithmic axis. In the commonest of the different varieties of logarithm paper put on the market by the firm of Schleicher & Schüll, the scale is such that the unit of the

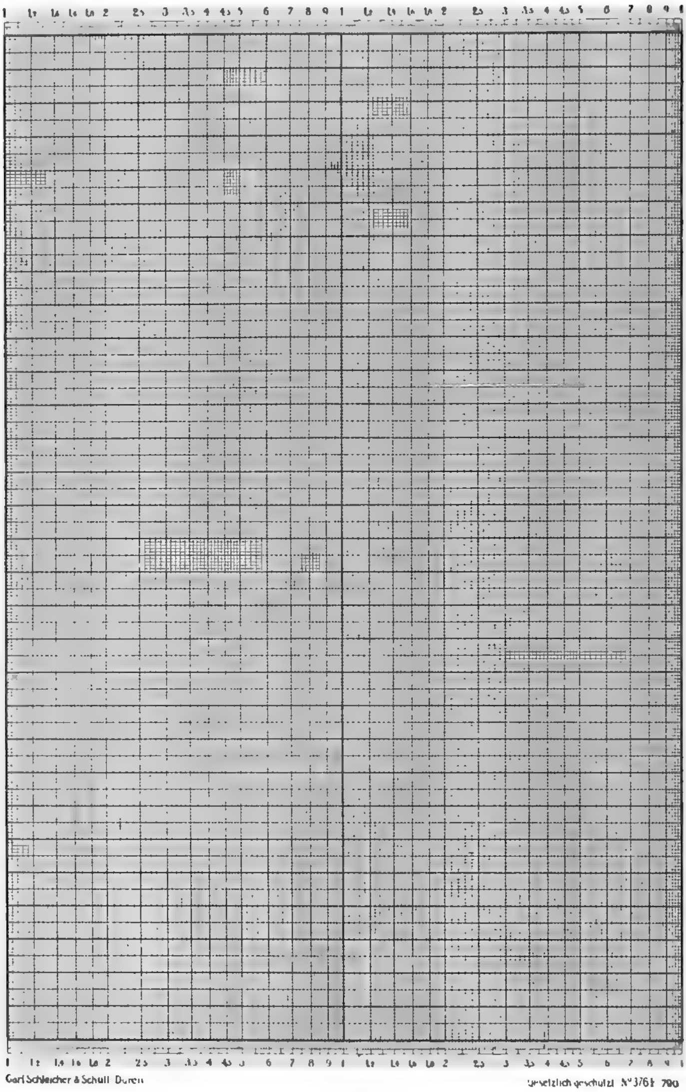


FIG. 2.

logarithm numbers, *i.e.* the distance between the two points marked 1 (really 10^n and 10^{n+1}) on the logarithm axis, is 250 mm. We thus get a so-called affine distortion of the diagram, just as in profiles exaggerated lengthwise. In the case under consideration, if we take the gradient from the graph as the quotient of two quantities measured in like units, we must multiply by ·004 or divide by 250. We must also notice that the logarithms

in equation (3) are natural, while common logarithms, for convenience sake, are used on the logarithm paper. Consequently we must divide the reduced gradient still further by the modulus of common logarithms, $M=0{\cdot}434$, or multiply it by $2{\cdot}303$. If we wish, then, to find the percentage increase in the examples previously mentioned, we must altogether multiply the k derived directly from the graph by $0{\cdot}004\times 2{\cdot}303\times 100=0{\cdot}92$. If it happened that the approximate straight line made an angle of 45° with the axis of abscissæ (time-axis), then we should obtain an increase in trade of $0{\cdot}92$ per cent. over that year which has been taken on the axis of abscissæ as equal to 1. Thus, for example, if the single years had been ranged on the axis of abscissæ

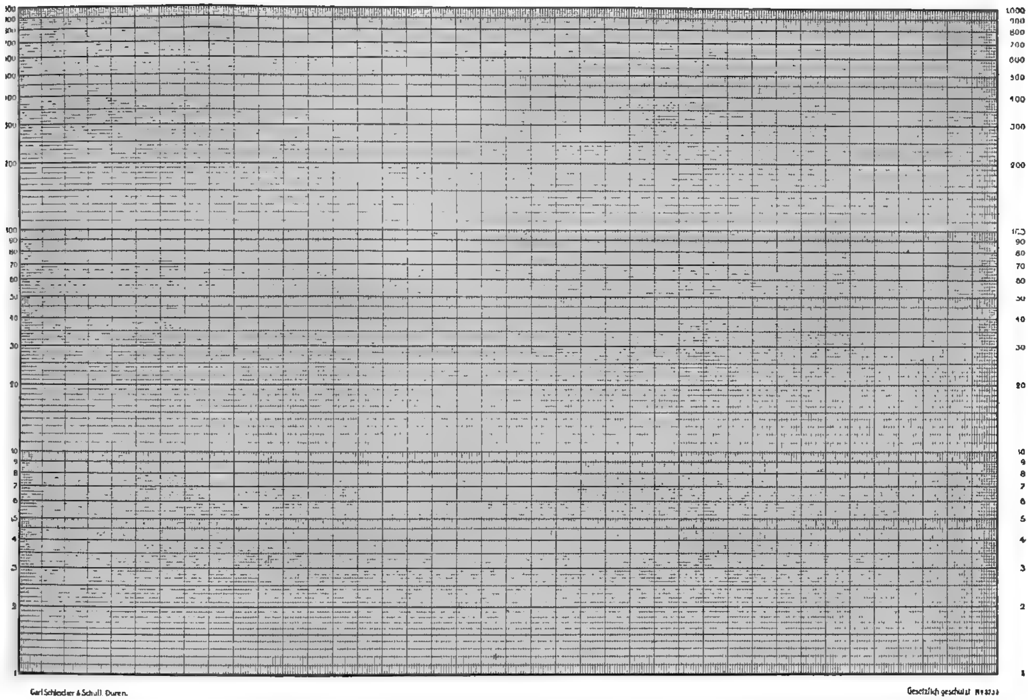


FIG. 3.

at intervals of 10 mm., we should have got the increase for $0{\cdot}1$ of a year, and the yearly increase of trade would then have amounted to $9{\cdot}2$ per cent. It is, of course, a matter of indifference what unit is taken for the "traffic" numbers, that is, whether we take as unit 100 or 1000 times the freight. We can, from the given "traffic" numbers, cut off as many figures (but an equal number from each) as we like, so that the remaining figures can be used in a range corresponding to the fineness of division of the scale.

We may add that the division just mentioned corresponds exactly with the lower division of a 25 cm. slide rule. The numbers on this logarithm paper also correspond partly to those on the slide rule. There is no number in the interval between 1 and 2, none in the further interval between 2 and 3, and so on. But by making some lines heavier than others, as on common graph-paper, which also shows no numbers, care is taken that, for example,

the points 1·5 stand out from the points 1·05, 1·10, 1·15, etc., and we do not easily make mistakes in plotting points. In this variety of paper it is possible to interpolate with three-figure accuracy in the closest part of the scale (between 9 and 10). If, therefore, the magnitudes of the "traffic" numbers in the above example were of such an order that none exceeded the number 10^6, we should cut off three places from all the numbers and work with a simple logarithm sheet, provided none of the stated numbers fell below 10^5. If this is not the case, we must use sheets in which several logarithmic divisions are ranged beside each other, exactly as in the upper scale of the common slide rule.

The example just taken, although it lies somewhat outside the interest of many readers, has been discussed in detail because it shows clearly the general principles involved.

Equation (1) represents on common graph-paper a curve which is convex to the x-axis. If we get a curve which has no maxima or minima and no contra-flexure and is concave to the x-axis, we arrive at the equation

$$y = \mathrm{K} \log_e \frac{x}{\mathrm{A}}. \qquad (4)$$

As we see, it represents the inverse of (1), if we put $\mathrm{K} = \frac{1}{k}$ and $\mathrm{A} = a$. Geometrically we then get the corresponding curve to (4), if we reflect the curve corresponding to (1) in the first octant line. Equation (4) on logarithm paper will likewise give a straight line, if we plot x along the logarithmic, and y along the linear axis.

Numerous applications, especially in geodetic problems, are got from the following consideration. If we regard the logarithmic axis as the x-axis and draw any continuous curve, then the area between the x-axis, the curve, and any two ordinates, *e.g.* those at x_0 and x_1, is the definite integral

$$\int_{x_0}^{x_1} y \frac{dx}{x}.$$

We immediately recognise the fundamental formula in barometric measurement of height, if we take x as the pressure at any place due to a column of air and y as the absolute temperature of the air at that place. The integral then represents (neglecting a constant) the difference of height between the two points in the air column at which the pressures x_0 and x_1 were measured.

This kind of calculation of barometrically determined differences of height is of extreme importance in balloon observations, where a continuous series of readings of pressure and temperature gives the varying altitude of the balloon and the maximum height to which it rises. As a rule such observations are calculated by the so-called "step" method, that is, we calculate from point to point by using either the ordinary barometric formula or the Babinet formula with height differences.

The advantage of logarithm paper is seen here, for by measuring the observed pressures on the logarithmic axis, and the absolute air temperatures as ordinates on the ordinary scale, we get by joining the isolated points a

broken line, and by means of a planimeter we can measure the area between the line, the x-axis, and the terminal ordinates. The area then gives on a certain scale the height reached by the balloon. The procedure is very easy and compact, and can also be modified by using, instead of absolute temperatures, the air temperatures read directly in Celsius degrees.

Another elegant application of logarithm paper in the determination of height differences obtained barometrically is got when we consider that the barometric formula can also be written in the form

$$\eta_1 = \eta_0 - \frac{x}{\mathrm{RT}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where $\eta_0 = \log_e p_0$ means the logarithm of the atmospheric pressure at a fixed place, T the mean (absolute) temperature of the air between two points, x the height above an initial point, and $\mathrm{R}=29{\cdot}3$ the gas constant. By a simple process, in which only straight lines are drawn, we get on the x-axis the different heights reached at each position of the balloon. Finally, we get a curve which represents the logarithm of the pressure as a function of the height. The single elements of this curve have on the logarithm paper the gradient $\frac{1}{\mathrm{RT}}$, and from this we easily get the construction of the curve. Here also we only require to draw straight lines, whose gradients we can get from a "ray" diagram, prepared beforehand for future use and numbered correspondingly. No determination of areas is here necessary. The integral resolves itself into a straight line on the x-axis. Finally, we can plot against the heights the observed temperatures as ordinates, and get without any calculation a diagram which represents the temperature as a function of the height. It is well known that one of the main problems of scientific balloon ascents is to investigate the changes of temperature with height.

Further applications are got when we have to investigate the change in some phenomenon or law whose mathematical expression is not known *a priori* and can only be ascertained in a purely empirical form.

In this case we frequently assume, if we are not dealing with a linear dependence or an approximation thereto, an expression of the form

$$y = a + bx + cx^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

to which, if desired, we can add a further term in x^3. In most cases we do not ask ourselves the question whether the assumption of an expression of the form (6) is at all justified by, or conforms to, the conditions.

The real reason for assuming such an artificially constructed law as the above rational function is that ultimately we have recourse in determining the constants appearing in the law to the method of least squares, which applies only to expressions of the form (6), or to such as can be brought under this form. And it is frequently the case that the application of these ingenious methods of approximation is in no wise conditioned by the exactness with which the law can be represented, but really by the fact that other methods for the determination of the constants are not at hand.

Before we assume in a particular case a law of the form (6), we ought every

time to ask ourselves whether the assumption of an exponential law is not equally justifiable. If we can answer yes to this question, we may have recourse to the latter law in consideration of the fact that in this (see equation 1) only two constants appear, and that the determination of these follows graphically if we only make use of logarithmically ruled paper. This graphical procedure will, of course, be only called into question when the application of the method of least squares does not furnish the required accuracy in our final results.

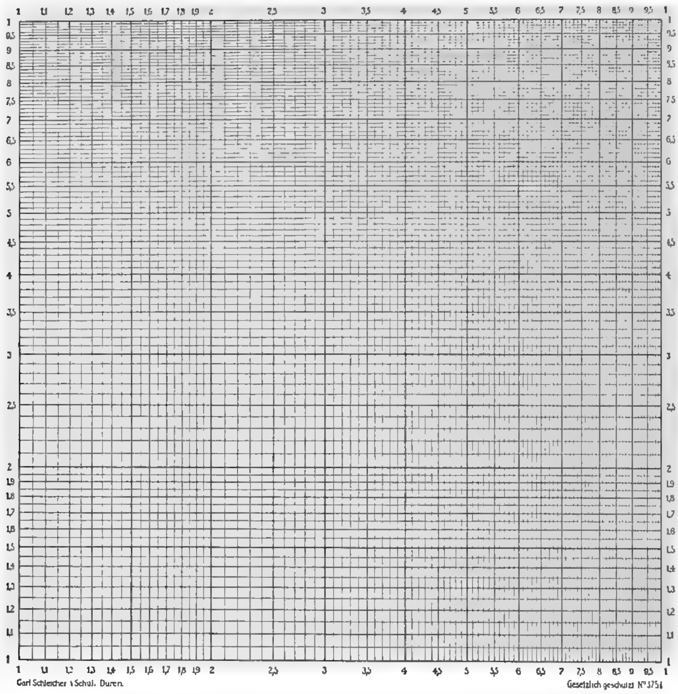


FIG. 4.

It is also customary in the calibration of hydrometric vanes to use the method of least squares, although here a graphical process would be suitable, and is, moreover, quite sufficient. In such calibrations we are concerned with finding out from observations conducted in the laboratory a formula which will give the velocity of flow, w, as a function of the number of revolutions, u, read off from the vane. As a rule we proceed thus: We lay off the u's as abscissæ and the w's as ordinates on ordinary graph-paper. If the plotted points be approximately on a straight line, we proceed no further, for we can then write w without further calculation as a function of u of the form

$$w=a+bu.$$

But if we get a curve—generally with its convex side to the axis of u—we must, if the logarithmic method is not possible, assume a relation of the form

$$w=a+bu+cu^2 \quad . \quad . \quad . \quad . \quad . \quad (7)$$

and determine a, b, c with the help of the method of least squares. It is more convenient, however, to assume an expression of the form

$$w = ae^{bu} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

which we can also write

$$\log w = \alpha + \beta u \quad . \quad . \quad . \quad . \quad . \quad . \quad (8a)$$

where by log we here mean the ordinary logarithm.

It will certainly not be disputed that the calculation of a table on the basis of equation (8) is much simpler than one calculated on the basis of (7).

The determination of the constants a and b in (8), or α and β in (8a), is very easily carried out with the help of logarithmic paper, and requires no further explanation after what has already been said. If we cannot obtain one straight line on the logarithm paper, we assume two formulæ of the form of equation (8), each of which holds for a definite region of u, the first perhaps for $u < 80$, the second for $u > 80$ revolutions.

Equation (8) will now be used, if the convex side of the curve is turned towards the axis of u; failing that, we assume an equation of the form (4), and in this case set off the number of revolutions u on the logarithmic axis. We should then take the point of division on the log axis marked 1 as $u = 10$, and the next point marked with 1 as $u = 100$.

It would be easy, and unnecessary, to multiply examples. It will, however, be simple for anyone who has occupied himself with similar problems and speculations, and is sufficiently acquainted with the principles involved, to apply in particular cases a suitable method from among those here mentioned.

The second kind of paper which is logarithmically divided in both directions is mainly of importance in the so-called "representation of isopleths." We can always represent a function z of two independent variables x, y,

$$e.g., \; z = f(x, y) \quad . \quad . \quad . \quad . \quad . \quad (9)$$

as a complex of isopleths. We can in general construct such isopleths by representing any pair of values x, y as a point in the co-ordinate plane, and describing it as the value of z corresponding to this pair of values. If we can calculate or assign the value to a sufficient number of points in a certain region of the plane, we can draw curves joining up the points which have the same values of z. We then have a complex of curves called isopleths, which collectively give a comprehensive picture of the change in the function $f(x, y)$, if we ascribe to each isopleth in the figure the corresponding value of z. We see that such isopleths are constructed in almost the same way as contour lines (isohypses), only in the case of the isopleths intersection of the curves is not in general excluded. It is, of course, not necessary for the construction of isopleths that the functional relation should actually be given by an analytical expression of the form (9). It is sufficient that for single corresponding discrete values of x and y the value of z is known, it may be from observation. We can easily see to what uses these isoplethic representations lend themselves. The value of the function corresponding to any pair of values x, y can be got at once from the diagram; they replace therefore tables with two columns of values, the calculation of which in most cases is detailed and lengthy.

Isoplethic representations, and especially such as are constructed, as mentioned above, from isolated observations, and require therefore interpolation, appear frequently in physical, meteorological, etc., applications. We can thus draw, for example, thermo-isopleths which give at a glance the mean temperature at a definite place, for each month of the year, and each hour of the day. To this class belong naturally the simpler cases of all isobar, isogone, and isohypse charts, traced for the surface of the earth.

More frequently it happens that the function $f(x, y)$ is given and well defined by a mathematical expression. The isopleths are then also given curves, to all of which equation (9) applies. The complex of isopleths arises

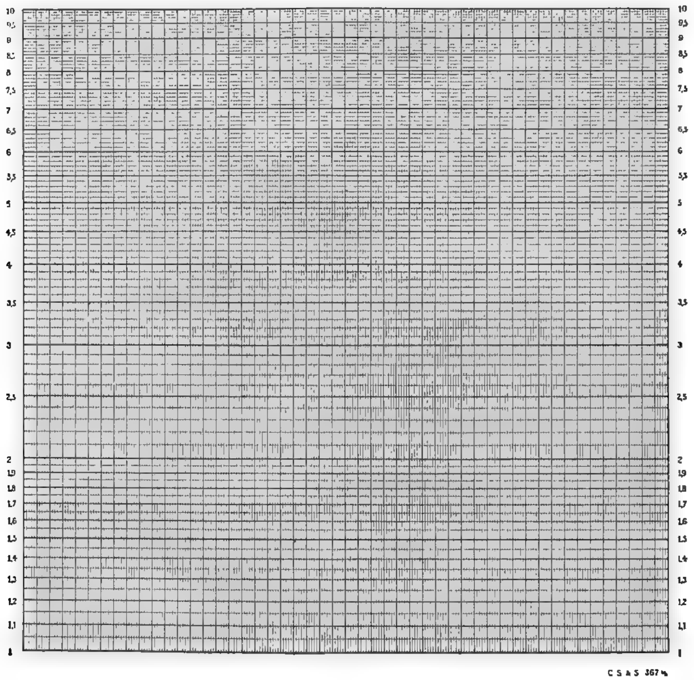


FIG. 5.

when we let the parameter z pass through definite discrete values. We shall, then, if the curves under consideration are simple ones, be able to construct them in accordance with geometrical rules.

The construction of isopleths will be specially simple in the case when these are straight lines—and it is this case which can be artificially introduced—if we use logarithm paper ruled logarithmically in both directions. If we draw on this paper a straight line with the current co-ordinates x', y', then to this straight line there will correspond on ordinary paper a curve with the current co-ordinates x, y, where $x' = \log x$, $y' = \log y$.

If the straight line on the logarithm paper has the equation

$$y' = \alpha + kx', \qquad . \quad . \quad . \quad . \quad . \quad (10)$$

it follows that $y = ax^k$, where $\alpha = \log a$. Here k may pass through all values between $-\infty$ and $+\infty$, and may in particular be put in the form of a vulgar

fraction. We can then say that the straight line on the logarithm paper represents a curve of the equation

$$y^m = Ax^n \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

according to the direction in which we take the straight line. If then the quantity A in (11) passes through all positive values, there correspond to all these curves parallel straight lines on the logarithm paper.

If we take, *e.g.*, $m=1$, $n=2$ (*i.e.* $k=2$), we get all the parabolas which pass through the origin and have their axes on the y-axis.

$m=1$, $n=-1$ (*i.e.* $k=-1$) gives all the equilateral hyperbolas whose asymptotes lie along the co-ordinate axes. The corresponding straight lines on the logarithm paper must therefore in this case be drawn so that they make equal intercepts on both logarithmic axes, *i.e.* pass through points of the same value.

All straight lines which on the logarithm paper make angles of 45° with both axes, *i.e.* are perpendicular to the preceding complex of straight lines, represent curves which would appear as straight lines through the origin on ordinary paper, for here $k=1$, $m=1$, $n=1$, and therefore

$$y = Ax.$$

The equation (11), which can also be written in the form

$$x^p y^q = A \quad . \quad . \quad . \quad (12)$$

shows very clearly the many different kinds of curves which all appear as straight lines on logarithm paper.

Here it appears almost superfluous to quote particular examples; we prefer to make the brief statement: Every function z of two variables x and y can be represented on logarithm paper by straight line isopleths, if it can be brought to the form

$$z = Cx^p y^q, \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

in which C, p and q are constants.

We should therefore be able with very little trouble to prepare a table of isopleths, from which one could at any time get the mean error M of an observation of weight P, if the mean error μ of the unit of weight is given, for it is

$$M = \frac{\mu}{\sqrt{P}},$$

and this expression is of the form (13); here $C=1$, $p=1$, $q=-\frac{1}{2}$. If these isopleths were drawn, the use of such a diagram would be, that we could now look for the given value of μ on the μ-axis, and get the point on the ordinate from it in which it is cut by the ordinate P. This point lies either on one of the given isopleths, and then the value of this isopleth gives at once the required M, or the point falls between two isopleths, and in this case must be interpolated by inspection.

On logarithm paper anyone can in two or three hours construct a table which completely replaces the 25 cm. slide rule. We only need to draw a complex of straight lines which go through the same numbers on both axes,

i.e. cut the x-axis at an angle of 135°; these straight lines, from what has been said above, represent hyperbolas and are the isopleths for the product

$$z = xy.$$

It must be noticed that the accuracy of such tables of isopleths can be made as great as we wish by taking the isopleths sufficiently close together, and we can do this by choosing from the varieties of logarithm paper one which has the unit of the logarithm scale sufficiently great.

Extremely interesting and, at the same time, fruitful applications are possible in the solution of certain equations of higher degree, which could only be solved with great difficulty if graphic methods were not available. Thus we may construct isopleths which give at a glance a root of the trinomial equation

$$x^m + ax^n + b = 0. \qquad \ldots \qquad (14)$$

For example, in solving the reduced cubic

$$x^3 = px - q \qquad \ldots \qquad (15)$$

we would start by constructing the two complexes of isopleths

$$y = px \qquad \ldots \qquad (16)$$
$$y = x^3 + q. \qquad \ldots \qquad (17)$$

The first appears on logarithm paper as a system of parallel straight lines, which lie at an angle of 45° with both axes. The complex (17) will now not be represented by straight lines, but by curved lines. The construction of a single isopleth of the complex (17) is, however, none the less easy if we now draw on logarithm paper the curve $y = x^3$, which appears as a straight line, and increase all the ordinates in such a way that the equation (17) is true. Here we can use an artifice which will be useful on future occasions. We begin by drawing the so-called logarithmic addition curve. This is given parametrically by the equations

$$\left.\begin{aligned} x &= \log t \\ y &= \log\left(1 + \frac{1}{t}\right) \end{aligned}\right\} \qquad \ldots \qquad (18)$$

which immediately show that the curve can be constructed by taking a series of chosen values of t and then plotting single points of the curve, using, of course, logarithm paper. We see at once that we can use such a curve to determine $\log(a+b)$, if $\log a$ and $\log b$ are given, for when

$$t = \frac{a}{b}$$

$$\log(a+b) = \log a + y,$$

provided y is the ordinate which belongs to

$$x = \log a - \log b$$

in the addition curve.

We can then find $\log(a+b)$ immediately from the curve, without requiring to draw any lines whatever, or making use of a table of logarithms.

It is now seen that the curve of equation (17) is easy to construct for a definite value, and so for any other positive or negative value, of q.

We have now two complexes of isopleths, one of which has the coefficient p, the other the absolute coefficient q, as parameter. A definite isopleth p intersects a definite isopleth q in a point whose ordinate y gives immediately a positive root of equation (15). A possible negative root of equation (15) may, of course, be found by finding the roots of the equation

$$x^3 = px + q \quad . \quad . \quad . \quad . \quad . \quad . \quad (14a)$$

in the same way, *i.e.* from the same diagram of isopleths.

We can see that this elegant process may be used without any difficulty in the solution of any equation of the general form (14), and that m and n may be any numbers whatever, integral or fractional, positive or negative.

(2) Exhibits of Ruled Papers. By Messrs Schleicher & Schüll

1. Squared Papers.
    In inch and centimetre sheets of various sizes.

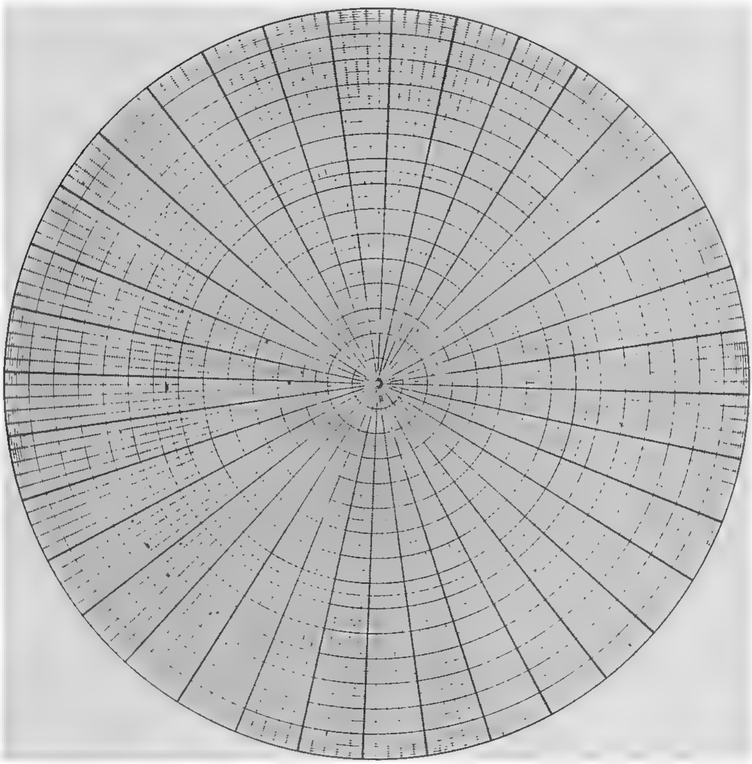


Fig. 6.

2. Logarithmic and Semi-Logarithmic Papers.
    In sheets of various sizes ; and with instructions for using.
3. Polar Degree Paper.
    Diameters of circle 10 cm. and 30 cm.

    Protractor Papers.
    Diameter of circle 20 cm.

4. Triangular Papers.

Sides of triangle 20 cm. and 50 cm.

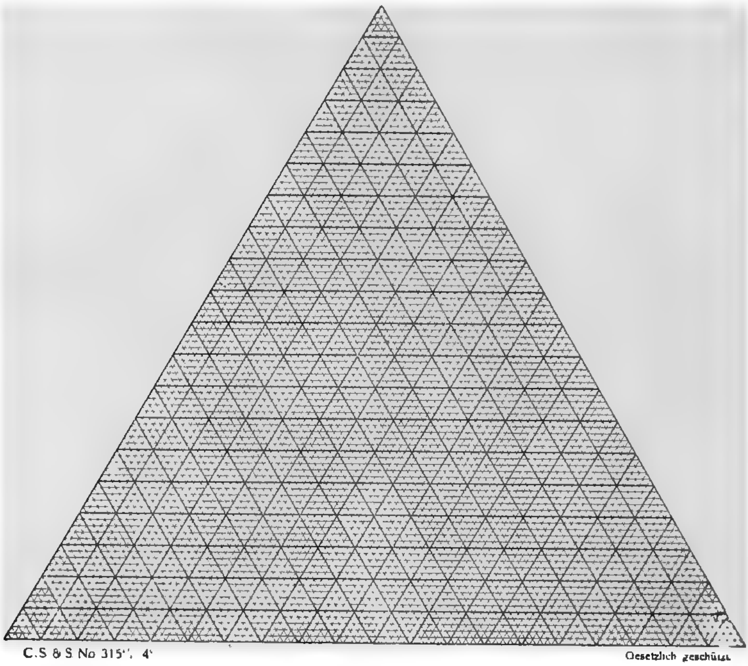


Fig. 7.

5. Charts.

(*a*) Year and ten-year charts in lengths of 71 cm.

(*b*) Charts with various rulings.

6. Ruled Tracing Papers.

In sheets 20 × 26 cm., of different colours and textures, and with various rulings.

7. Drawing Pads as follows:—

One pad each *Logarithmic Paper*, Nos. 365½, 367½, 375½, 376½, 373½.

One pad No. 315½ *Co-ordinate Chart Paper*, with triangular ruling.

One pad each Nos. 316½, 316½ : 30 do. with circular ruling.

One pad each Nos. 378½, 397½, *Harmonic or Sine Paper.*

One pad No. 318½ *Meteorological Chart Paper.*

One pad No. 317½ *Earthquake Chart Papers for seismological records.*

One pad No. 399½.

One pad No. 350½ *Drawing Paper Charts for annual reports.*

One pad each of Nos. 332½, 332½ : 20, 332½ : 24, 324½, 325½, 326½, 327½.

One book containing sample sheets of all our Sectional, Profile, Logarithmic, Sine, and Co-ordinate Papers.

(3) A Radian-Polar Paper. By E. M. Horsburgh, M.A.

In this paper the angles are set out in radians and decimals of a radian. There is a conversion scale on the border of the paper which converts radians to degrees, and conversely. In the first quadrant there are two semicircles on diameters subdivided decimally. These intercept on any ray lengths representing to scale the cosine and sine of its inclination, and hence the tangent may be obtained. Thus the values of the circular functions are shown on the

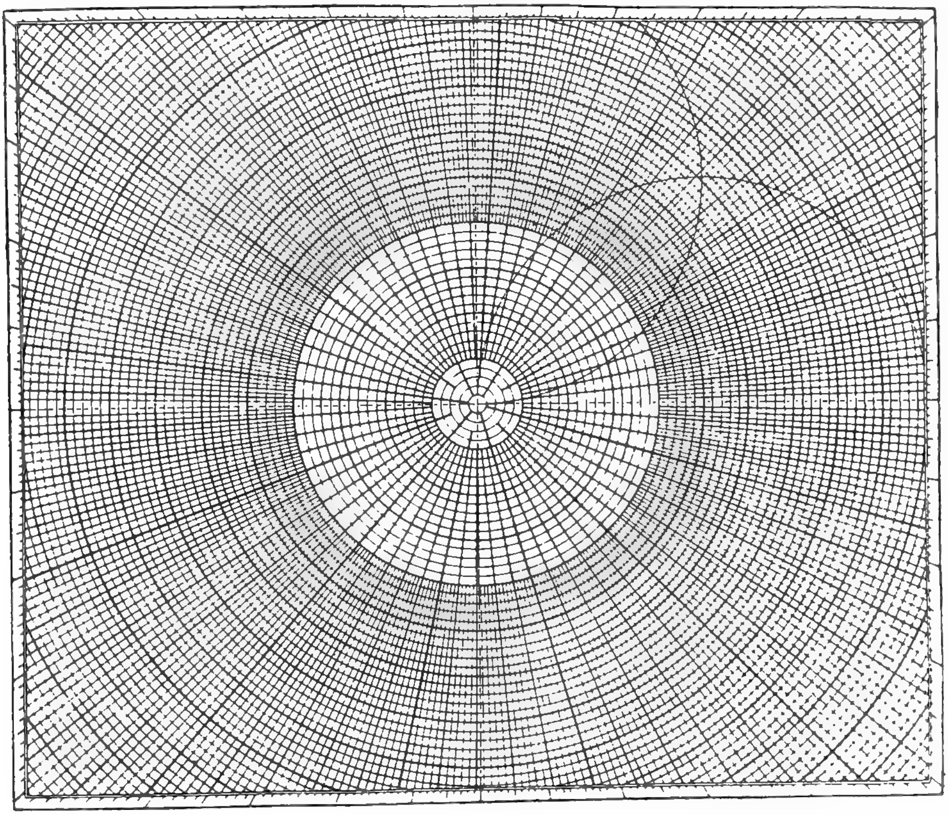


Fig. 8.

paper, which may simplify the plotting of polar graphs. It is evident, too, that the various processes of graphic arithmetic may be illustrated by this paper, while reciprocals are given by the semicircle intercepting any ray.

The paper is printed in two forms: fig. 8 represents the complete circle, and fig. 9 the first quadrant.

When a graph has been drawn, its scale may be increased or diminished in sectors, as may be convenient for calculation. The important integrals $\int rd\theta$ and $\int r^2d\theta$ may then be evaluated approximately, frequently with considerable accuracy, the latter as an area, the former as a "departure," or as the sum of the mean radii vectores of the small sectors represented on the paper, multiplied by the radian value of the angle of the small sector.

It is evident also that this paper has many of the advantages of squared paper, as it may be used for interpolation, and for the graphical correction of errors of observation. It also gives a means of representing directly the

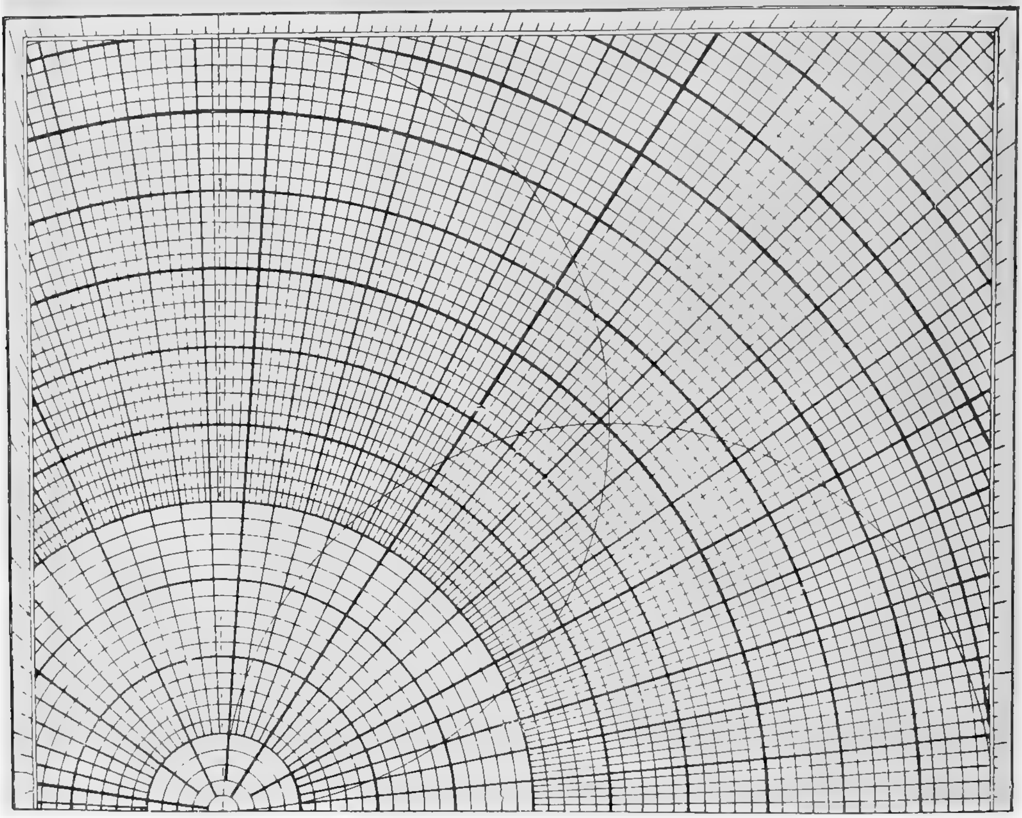


FIG. 9.

results of experiments dealing with angular displacements, and, while intended primarily to simplify graphical calculations involving the methods of polar co-ordinates, it may also be used to give to beginners some elementary ideas in trigonometry.

II. Collinear-point Nomograms (Nomogrammes à points alignés).

Exhibited by Professor M. D'OCAGNE, École Polytechnique, Paris. (Translation by A. W. YOUNG, M.A.)

THESE four nomograms are constructed according to the method of collinear points (*la méthode des points alignés*), the principle of which was first described in 1884 by Professor d'Ocagne in the *Annales des Ponts et Chaussées* (2^{e} Sem. p. 531), and which he has since developed in his works:—*Nomographie* (Gauthier-Villars, 1891), *Traité de Nomographie* (Gauthier-Villars, 1899), *Calcul graphique et Nomographie* (Doin, 1re éd., 1908; 2^{e} éd., 1914).

The unknown quantity which is to be determined is read off on the nomogram by means of a thread stretched between points on certain scales, these points being selected in accordance with the data of the problem.

I. Nomogram for the Calculation of the Cross-section of Embankments and Cuttings in Road Construction

This nomogram is a combination of three simple nomograms corresponding respectively to the case of an embankment (*Remblai*), to that of a cutting (*Déblai*), and to a complementary term (*Terme complémentaire*) for use in the case of a cutting with an embankment. All the particulars about the construction of this nomogram are given in *Leçons sur la topométrie et la cubature des terrasses*, by Professor d'Ocagne (Gauthier-Villars, 1904), pp. 176 to 182. The scales used are all *logarithmic*.

The reading thread, stretched between the point corresponding to the measure of the depth (*Côte en remblai, Côte en déblai*) and the point corresponding to the measure of the slope of the land (*Déclivité du terrain*), cuts, on each partial nomogram, the three other scales in points giving the breadth of the roadway (*Emprise*), the length of the slope (*Talus*), and the area of the cross-section of the excavation (*Surface*).

In the case of an excavation which is partly a cutting and partly an embankment, we first go to that one of the partial nomograms (*Remblai* or *Déblai*) on which the stretched thread does *not* cut the vertical barrier along which is written the word "*Arret.*" At the same time, however, the thread cuts the barrier drawn in broken line bearing the words "*Terme complémentaire.*" This warns us that for the *surface* we must add to the number read on the first nomogram the quantity which is furnished by the nomogram of the *Terme complémentaire*.

II. Nomogram for the Approximate Determination of the Span of a Catenary

In a note published in the *Annales des Ponts et Chaussées* (1910, 4[e] fasc., p. 114), Professor d'Ocagne has shown that, in questions arising concerning the span of bridges, we may, for an arc (symmetrical about the axis of y) of a transcendent curve defined by such a series as

$$y=\frac{a_2}{2!}\frac{x^2}{p}+\frac{a_4}{4!}\frac{x^4}{p^3}+\dots,$$

substitute the *osculating conic* at the origin of co-ordinates, namely,

$$3a_2^3x^2+a_4y^2-6pa_2^2y=0.$$

In the case of the *Catenary* ($a_2=a_4=1$), the equation becomes

$$3x^2+y^2-6py=0.$$

It is this equation that is represented by the nomogram, a full explanation of the construction being given in the note cited above (p. 126).

In a recent note appearing in the same Journal (1914, fasc. i. p. 160), the author has pointed out that the same nomogram may still be used in the case of the *catenary of uniform strength* ($a_2=1$, $a_4=2$), and in that of the *cycloid*

($a_2=\frac{1}{4}$, $a_4=\frac{1}{16}$). For this, however, if we are to make use of the same scale for (x), we must multiply for the first case the quantities (y) by $\frac{1}{\sqrt{2}}$ and the quantities (p) by $\sqrt{2}$, and for the second case the quantities (y) and (p) each by $\frac{1}{2}$.

III. Nomogram for the Solution of the General Equation of the Third Degree

This nomogram gives (within the limits of graduation) the positive roots of the equation

$$z^3+nz^2+pz+q=0;$$

the moduli of the negative roots may be obtained as the positive roots of the transformed equation in $-z$.

The theory of this nomogram is given in detail in each of the three works of Professor d'Ocagne above mentioned: *Nomographie* (No. 46), *Traité de Nomographie* (No. 125), *Calcul graphique et Nomographie* (No. 73).

The mode of use is contained in the following precept: *Stretch a thread between the points marking p and q on the vertical scales; the thread will cut the curve associated with the number n in certain points; the quantities z signified by the verticals passing through these points are the roots of the equation.*

IV. General Nomogram of Spherical Trigonometry

If, knowing any three of the six elements of a spherical triangle, we wish to calculate the other three, we can do this by means of the single formula

$$\cos a=\cos b \cos c+\sin b \sin c \cos A,$$

where we may make any permutation we please among the elements, applying, if necessary, the properties of the supplementary triangle. The nomogram representing this formula may thus be utilised for all the cases of solution of spherical triangles.

On this nomogram, of which the theory is given in the *Traité de Nomographie* (No. 124), the scale on the lower horizontal axis is that of a, and the scale on the upper horizontal axis is that of A. The point (b, c) is at the intersection of the ellipse associated with the number b and the ellipse associated with the number c, when we take into account the following rule: there being associated with each ellipse two numbers b and c, supplementary to each other, that particular (b, c) is taken which is in the quadrant to the left or to the right, according as b and c are on the same side or on different sides of 90°.

All ambiguity being thus avoided, the mode of use of the nomogram is given in the simple proposition: *the thread stretched between the points* (a) *and* (A) *passes through the point* (b, c).

Like the preceding, this nomogram furnishes an example of the representation by the method of collinear points of an equation with four variables, to which it would have been impossible to apply the method of intersection, since we are unable to group two of the variables into one member and the other two into the other.

III.

(1) Exhibit of Computing Forms used in Harmonic Analysis, from the Mathematical Laboratory, University of Edinburgh.

(2) Exhibit Lent by the Director of the Meteorological Office, London, S.W.

(i) Computing forms for pilot balloon work.
(ii) General Strachey's slide rule for determining heights of clouds from photo-theodolite observations.

Section I

MATHEMATICAL MODELS

I. Mathematical Models. By Professor Crum Brown, D.Sc., LL.D.

Mathematical models have the same use in solid geometry as diagrams have in plane geometry. They are helps to the imagination. They need not be, they cannot be, perfectly accurate representations of the objects about which we reason; they serve their purpose if they enable us to see these objects accurately with the mind's eye, and so reason correctly about them. All the same, in making a model, as in drawing a diagram, care should be taken to avoid inaccuracies when this is possible. We cannot prove a proposition by measuring lines or angles in the diagram or model; when we make such measurements our object is to test the accuracy of the representation. We should not think of obtaining the value of $\sqrt{3}$ by making a model of a cube, and measuring the length of a body diagonal; yet, if we make such a model, we should see to it that the four body diagonals are sensibly equal.

Some inaccuracies, arising from the nature of the materials used, are unavoidable; one of these may be seen in the model of the "half-twist" surface. To make this model perfect, the plate of which it is formed should have no thickness; as this cannot be, we should make it as thin as possible, consistently with the necessary strength. In the model shown the plate might, with advantage, have been considerably thinner.

Models may be made of many different materials. Very good models of crystal forms, and of other polyhedra, have been made of wood. Surfaces of rotation can, of course, be easily turned at the lathe, the work being guided by means of calipers, and a templet representing a plane section containing the axis. Other curved surfaces have been cut in wood, using cardboard cut along lines representing plane sections of the curved surface as templets to guide the cutting. By means of such templets models can be made in a plastic material, such as clay or wax, and then cast in plaster. Such a cast may be painted, and lines, representing plane sections of the surface, may be drawn upon it, either temporarily with lead pencil, or permanently in oil colour. The curved surface modelled may be representative of an equation with three variables, such as $f(x, y, z) = 0$, or it may represent the relations, experimentally found, between three physical properties of a substance. Of this kind is Professor James Thomson's model showing the results of Professor Andrews' determinations of the relation of temperature, pressure, and volume of a constant mass of carbonic anhydride. In this model x is temperature, y pressure, and z volume.

Models of polyhedra can be cut out of wood or ground out of solid glass, but for the amateur model-maker the best material is cardboard. On this is drawn what the Germans call a "Netz"; what this is will be understood by looking at the examples exhibited. The cardboard is then cut by means of a sharp knife against a steel straight-edge, quite through along the boundary lines of the Netz, and a little more than half through along the internal lines. The cardboard can then be folded up, bending it where it is half cut through, so as to form the polyhedron. Each solid angle is then secured with a drop of sealing-wax; seccotine is applied, by means of a fine hair pencil, to the edges; when this is hard, the sealing-wax is carefully removed with a sharp knife, and seccotine applied to the parts of the edges thus exposed. The model may then be painted. The cardboard models exhibited are mounted on stands, by fixing a brass tube through holes in opposite faces or opposite solid angles, the tube passing through the centre of the figure. The mode of making the models of the higher species of polyhedra will be described under that head.

Very useful models can be made of wire, string, or thread, each string being fixed at its ends to solid supports, which may be of wood or of metal. All ruled surfaces can be illustrated in this way. Among the examples shown are ruled and developable surfaces, and, in particular, the ruled quadric surfaces, Dr Sommerville's model of the projection, on three-dimensional space, of a four-dimensional figure, and the group of models of this nature exhibited by Professor Steggall, as well as his deformable wire models.

Some models of curved surfaces are shown, in which parallel plane sections are represented by sheets of paper interlocked so that the distance between neighbouring sheets can be varied, thus varying the constants in the equation representing the surface.

Kinematic models show how the motion of one point in a system is related to that of the other points; thus, for instance, how circular motion of one point produces simple harmonic motion of another. Lord Kelvin's tide-calculating machine is a kinetic model showing how several simple harmonic motions can be combined. Indeed, every machine is a kinematic model, for, besides doing its own work, it illustrates some kinematic relation.

Seven Groups of Models exhibited by Professor Crum Brown

I. *Plaster Models* (1) *of the Surface* $z=3a(x^2-y^2)-(x^3+y^3)$; (2) *of the Surface* $2z=a^2(x^2+3y^2)-(x^4+6x^2y^2+y^4)$

In each of these models lines are drawn representing plane sections. It may be noted that the section of (2), the biquadratic surface, by a horizontal plane through the two points where z is a minimax, consists of two ellipses, the major axis of one of which coincides with the minor axis of the other. The models were made for the late Professor Chrystal to illustrate his lectures on equations.

II. *A Model of the "Half-Twist" Surface*

The "twist surfaces," of which this is a case, stand in the same relation to the helicoid surface as the anchor-ring does to the cylinder. In the helicoid the generating line, at right angles to the axis, rotates about the axis

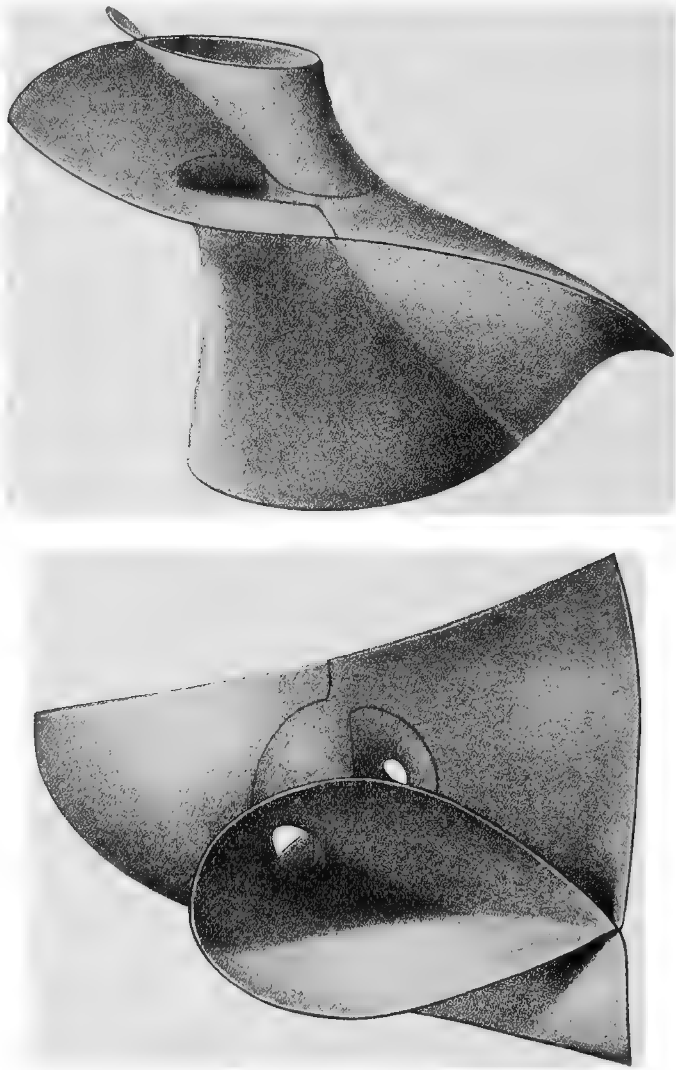


FIG. 1.—Model of the "Half-twist" Surface.

as the point of intersection moves along it. In the twist surfaces the generating line is always at right angles to a fixed circle, and rotates about the tangent to the circle at the point of intersection, as the point of intersection moves round the circle. The species of twist surface is defined by the ratio of the angular motion of the generating line to that of the point of intersection. In the particular case illustrated by the model, the generating line turns through

two right angles, while the point of intersection makes one whole revolution; that is, the rate of angular motion of the generating line is one-half of that of the point of intersection.

The idea of making such a model was derived from the "one-sided surfaces" exhibited by Professor Tait, formed by gumming together the ends of a strip of paper, after giving it half a turn about its axis. Such a strip has only one side and only one edge, or, perhaps more accurately, its two sides are continuous, and its two edges are continuous. If such a strip is very narrow, and if it is so arranged that its central line is a circle, it may be considered as a portion of a "half-twist" surface. Without entering into any detailed mathematical discussion of the surface, there are some points of interest which may be indicated. A straight line passing through the centre of the circle, and at right angles to its plane, obviously lies wholly in the surface, as every generating line cuts it. We may call this line the axis of the surface. Every plane through this axis contains two generating lines; the intersections of these pairs of generating lines lie in a straight line touching the circle, and inclined at an angle of $\frac{\pi}{4}$ to its plane. The surface therefore intersects itself in this straight line. It is obvious that the surface has "helicoid asymmetry"; as, for each sense in which the point of intersection may rotate, there are two senses in which the generating line may rotate. This gives four forms, which obviously coincide in pairs.

III. *Group of Six Models illustrating the Partition of a Cube into Six Equal Tetrahedra without making New Corners*

There are four different tetrahedra of equal volume which can be cut out of a cube without making new corners. One face which occurs in all the four is the half of a face of the cube; its sides are a face diagonal and two edges of the cube. We may take this face as the base of each pyramid, the summit being one of the four corners of the cube not in the plane of the base. There are thus four forms, and these are obviously equal. The volume of each is $\frac{s^3}{6}$, where s is an edge of the cube, and of course s^3 the volume of the cube. Models of these tetrahedra are shown marked Δ, I, Γ, and L. Their faces are as follows: Δ has three contiguous half faces of the cube; its fourth face is an equilateral triangle whose side is a face diagonal of the cube. I has two scalene triangles, whose sides are an edge of the cube, a face diagonal, and a body diagonal, a half face of the cube and an equilateral triangle as in Δ. L and Γ are enantiomorph, *i.e.* the one is the same as the mirror-image of the other. Their faces are two half faces of the cube, and two scalene triangles, as in I. The scalene triangles of Γ are enantiomorph to those of L; and the two scalene triangles of I are enantiomorph, the one being the same as those of Γ, and the other the same as those of L. The notation is intended to indicate the number of half faces of the cube in each tetrahedron by the number of straight lines in its symbol, and Γ and L are chosen for the two enantiomorph forms, because these symbols are also enantiomorph.

In a cube built up of those tetrahedra the number of those of the form I is always equal to that of the form Δ, and one of the one is always contiguous to one of the other, the two tetrahedra forming together a figure which we may call IΔ. It is an oblique square pyramid, the base being a face of the cube, and the apex one of the corners of the cube not in the base. It has the same form as an L and a Γ joined together by a scalene triangle of each. By adding to it either a Γ or an L, a half cube is formed. From these data we can deduce the number of ways in which a cube can be built up. These are shown in the models exhibited.

IV. *Group of Models of the Regular Solids, and of Forms related to them*

A higher species is obtained from a polyhedron by producing its faces until they meet again. It is obvious that there can be no higher species of the tetrahedron, for in it every face already cuts every other. The second species of the cube consists of three intersecting square prisms, the faces of which may be said to intersect again at infinity. The second species of the octahedron consists of two intersecting tetrahedra. The third species is one of infinite volume, consisting of six intersecting rhombic prisms. In the model these prisms are cut off irregularly, to indicate that they are supposed to extend indefinitely.

Counting the first, and excluding the forms with prisms, there are four species of the regular dodecahedron, and eight, in the systematic order of development, of the icosahedron. By development in systematic order is meant the formation of the second species by producing the faces of the first until they meet again, and of the third in the same way from the second, and so on. These four species of the dodecahedron, and eight of the icosahedron, are all shown in models. In the case of the higher species of the dodecahedron the models are cut along the plane of one of the faces, so that the intersection of faces in the interior can be seen. The four species of the dodecahedron are all regular. Their faces are regular polygons; the faces of the first and of the third are ordinary pentagons, those of the second and of the fourth are pentagons of the second species—the so-called pentacle or "Drudenfuss."

The fifth species of the regular dodecahedron consists of fifteen intersecting rhombic prisms. A model is shown illustrating the development of this fifth species from the fourth. In the model one-third part of the complement (five of the fifteen prisms) is represented.

Of the eight species of the icosahedron, derived in systematic order, only the first and the seventh are regular; their faces are equilateral triangles. The third species is of special interest. It looks exactly like a set of five independent and intersecting octahedra, and the model is coloured to show this. But a closer examination makes it clear that this is not so. For five independent octahedra would have 5×8, that is to say, forty faces; but this is an icosahedron, and therefore has only twenty. And looking at a face, we see that it is formed of two intersecting equilateral triangles, as shown in the diagram annexed to the model. Now, one of these triangles is a face of one of the five octahedra, the other of another; the common part belongs to both of these octahedra, and it is because this common part is hidden that the true

nature of the form is not at once seen. It seems to be regular and discontinuous; it is really continuous and not regular.

There is an interesting form derived from the icosahedron, but not in the systematic order. The faces of the ordinary icosahedron (the first species) can be divided into five groups of four, the four faces in a group being related to one another as the faces of a tetrahedron. If, then, the faces of one group are produced they meet and form a tetrahedron; and so with the other four groups. This aggregate of five tetrahedra is a fifth species of the icosahedron, for to get from the outside to the centre we must pierce the surface five times, each tetrahedron once. These five tetrahedra are really independent; no part of a face is common to two of them. The form is regular, its faces are equilateral triangles, but it is discontinuous. There are two distinct ways in which the faces of the icosahedron can be divided into five groups of four, and each of these ways gives rise to a set of five intersecting tetrahedra, these two sets being enantiomorph. In the models each has attached to it a model of an icosahedron with the faces coloured to show the five groups. We may call this the asymmetric fifth species of the icosahedron.

There are two (not regular) solids closely related to the regular polyhedra—the rhombic dodecahedron and the rhombic triacontahedron. The first has its twelve faces corresponding in position with the edges of the cube and of the octahedron; the second has its thirty faces similarly related to the edges of the regular dodecahedron and of the icosahedron. These relations are shown in the models (*a*) of a cube and an octahedron intersecting, and (*b*) of a dodecahedron and an icosahedron intersecting, in which a pair of normally intersecting edges represents the diagonals, in the one case of a face of a rhombic dodecahedron, in the other of a face of a rhombic triacontahedron. These solids also have higher species. Models are shown of all the higher species of the rhombic dodecahedron—the second, third, fourth, and fifth. The fourth has four regular hexagonal prisms, and the fifth consists of four pairs of coaxal triangular prisms and three square prisms all intersecting. In these models the prisms are represented as broken off as in the third species of the octahedron. Of the triacontahedron, models are shown only of the first, second, third, fourth, and fifth species. The fifth species is interesting as being an aggregate of five intersecting but independent cubes. In all these models of the triacontahedron and its higher species, the five groups of six faces, each of which forms a cube, are distinguished by colour.

In making models of the higher species, in most cases the best way is to prepare a model of the first species, and convert it into a model of the second species, by adding to each face what may be called the complement; and from the second to make the third in a similar way, and so on, in what we have called the systematic order. The forms of the faces of the several complements may be obtained from the complete plan of a face of the polyhedron. This is made by taking a face as the plane of reference—the plane of the paper—and drawing on it the straight lines in which the plane of each other face (except, of course, the parallel plane) cuts this plane of reference. Some of these complete plans are shown.

Kepler seems to have been the first to describe and discuss the higher species of the regular solids (1619). The subject was dealt with by A. L. F.

Meister (1771). But these early notices fell into oblivion, and Poinsot, in 1809, rediscovered these forms. They have since been discussed by a considerable number of mathematicians, among whom may be mentioned Cauchy, Bertrand, Cayley, Wiener, Brückner, and Haussner.

V. *Interlacing Surfaces*

The simplest form of the interlacing surfaces as spread upon a plane is illustrated in fig. 2. It will be seen that we have here three sheets, differently shaded so as to distinguish them to the eye, but otherwise quite similar.

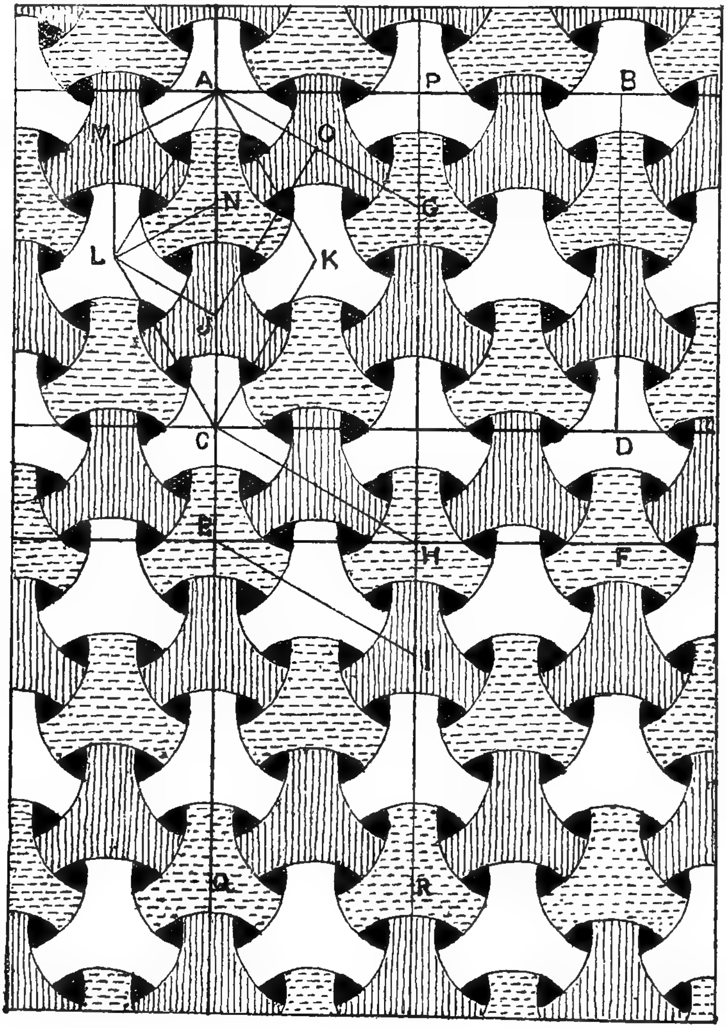


Fig. 2.

Each sheet is perforated by equal circular holes so arranged that any three neighbouring holes in the same sheet have their centres at the apices of an equilateral triangle. The radius of the holes must not be greater than half the distance between the centres of two neighbouring holes, otherwise the

sheet would be cut into separate pieces; and must not be less than one-third of the said distance, otherwise there would not be room for the neck between two holes in one sheet to pass without crumpling through the chink caused by the overlapping of the holes in the other two sheets. In the figure the radius of the holes is about two-fifths of the distance between the centres.

The complex of three sheets is, as will be seen by inspecting the figure, a case of what Professor Tait calls *locking*. No two sheets are *linked* together; if any one sheet be abolished the other two come apart. Each sheet lies wholly above one of the other two, and wholly below the other.

The analogy of this complex to what we may call the Borromean[1] rings will be seen at once. In the Borromean rings figured below (fig. 3), each ring lies wholly above one of the other two, and wholly below the other, so that

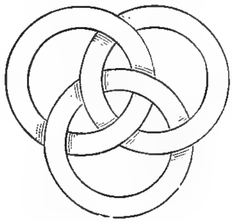


FIG. 3.

while all are inextricably locked together, no two are linked, and if any one is abolished the other two come apart.

The complex of sheets may be applied to other surfaces besides the plane. Two other surfaces, viz. the cylinder and the anchor-ring, will be considered here.

To apply the complex to a cylinder, or to clothe a cylinder with the interlaced sheets, we must cut the complex by two parallel lines, and roll up the strip thus cut out so that the two edges shall join and form what may be called the seam. But there must not be any peculiarity at the seam; the pattern must run through the seam without any discontinuity; therefore the two parallel lines must cut the complex in the same manner, so that each part of a hole divided by one line may find its exact continuation at the seam when the strip is rolled up.

There are, of course, an infinite number of ways in which such a strip may be rolled up into a cylinder. But in whatever way the cylinder is formed, if we cut it along a generating line, and unroll it, we may take the parallel edges of the flat strip as the two lines defining, on the plane complex, the particular cylinder. We may move these two parallel lines, parallel to themselves, retaining their distance from one another, in any way, and they will still represent the same cylinder, because we may form this flat strip by cutting the cylinder through any generating line. Two parallel lines will therefore represent a cylinder if the points in which they intersect a line at right angles to them are always similarly situated in reference to the complex.

The number of cylinders is obviously infinite, but they may all be grouped

[1] The interlocked rings shown in fig. 3 occur in the armorial bearings of the Italian family Borromeo.

under two genera. For a part of a hole, cut off by one of the parallel lines, may, at the seam, find its continuation in a part of a hole, either, first, in the same sheet, or, second, in one of the other sheets.

In the first case we have three distinct sheets locked together. In the second we have only one sheet wound three times round the cylinder, and knotted. When we have three independent sheets we can colour or shade them independently, each having its own colour or shading; but when there is only one sheet this is not possible. In this case the only way of distinguishing the layers is by varying the colour, or shading, continuously as we go round the cylinder, so that after three turns we come back to the colour or shading with which we started. This has been done in the models exhibited.

We have assumed that the complex is flexible. We shall now assume that it is also extensible, so that we can draw it out in any particular direction, and make the circular holes into ellipses. We shall assume that any deformation may be produced without affecting the character of the complex as long as the topological relation of the layers is preserved. This extension is not of any use if we confine ourselves to cylinders, for there is no topological change produced by twisting a cylinder. The meaning of the extension will be seen when we come to apply the complex to an anchor-ring.

An anchor-ring can be made out of a cylinder in two ways. We may cut the cylinder by two planes at right angles to the axis, and bend the part thus cut out round so that its axis becomes the core of the anchor-ring; or we may cut the cylinder, and then widen out the two ends, and bend them over so that they may unite and form a seam, not about the core, as in the last-mentioned case, but about the axis of the anchor-ring. An anchor-ring has thus two seams—one a circle with its centre in the axis, and one a circle with its centre in the core—and it can be reduced to a cylinder, either by cutting the first, and, if we may coin the word, "unflyping," or by cutting the second, and unbending.

We see, then, that just as a cylinder can be represented by two parallel lines, so an anchor-ring can be represented by a parallelogram. The condition here is, that the parallel sides of the parallelogram cut the complex in precisely the same way. Such a parallelogram will in general represent two anchor-rings; we must therefore indicate which of the two pairs of parallel lines represents the seam about the axis, and which the seam about the core. To transfer from this plane plan to an actual anchor-ring—that is, to make a model such as those exhibited—we have only to remember that the four corners of the parallelogram represent the single point in which the two seams intersect; that the one pair of parallel sides represent the one seam, the other the other; and that lines parallel to these pairs of sides are to be measured on the anchor-ring, in the one case along the circumference of a circle about the axis, in the other case along the circumference of a circle about the core.

As there are two genera of cylinders, one knotted and one locked, so there are four genera of anchor-rings: 1st, locked about the axis and locked about the core; 2nd, locked about the axis and knotted about the core; 3rd, knotted about the axis and locked about the core; 4th, knotted about the axis and knotted about the core. Of these, only the first, which is not knotted at all, consists of three distinct sheets; the second is reduced to a locked

cylinder by cutting it along a seam about the axis, to a knotted cylinder by cutting it along a seam about the core; in the third, these relations are reversed; in whichever way the fourth is reduced to a cylinder, a knotted cylinder is produced.

It is worthy of notice that anchor-rings of the fourth kind have necessarily "helicoidal asymmetry." A ring of this kind is necessarily enantiomorph to its mirror-image.

In fig. 2 the lines AB, CD represent a locked cylinder; AB, EF a knotted cylinder; AQ, PR the smallest locked cylinder; CD, EF the smallest knotted cylinder; the parallelogram ABCD an anchor-ring of the first kind; AGCH one of the second kind, if AC and GH represent the seam about the axis; one of the third kind if these lines represent the seam about the core; AGEI one of the fourth kind; AKLC the smallest ring of the first kind, with one hole in each sheet; AOLJ the smallest ring of the second (or of the third) kind, with two holes altogether; MALN the smallest ring of the fourth kind, the smallest ring indeed of any, having only one hole altogether.

We have hitherto considered the complex as composed of *perforated* sheets locked together, or of a *perforated* sheet knotted; but there is another way in which it may be imagined.

We saw that the smallest circular hole had a radius of one-third of the distance between the centres of two neighbouring holes in the same sheet; but we can make the hole smaller if, instead of making it circular, we make it hexagonal. There is then no waste space; every part of the complex is composed of two layers, one over the other. Now we may suppose this hexagonal boundary to be, not the edge of a hole, but a line of intersection, where the surface, instead of ceasing, disappears between the two other sheets.

The knitted model exhibited illustrates this form of complex.

VI. *Plaster Cast of Professor James Thomson's Model, illustrating Modes of passing from the Gaseous to the Liquid State.* Lent by Professor CRUM BROWN

Lecture by Professor Andrews on "The Gaseous and Liquid States of Matter,"[1] Royal Institution of Great Britain, 2nd June 1871.

[1] These different modes of passing from the gaseous to the liquid state are admirably illustrated by a solid model constructed by Professor James Thomson, which was exhibited at the lecture. I have been favoured by Professor Thomson with the following description of this model:—

"The model combines Dr Andrews' experimental results in a manner tending to show clearly their mutual correlation. It consists of a curved surface referred to three axes of rectangular co-ordinates, and formed so that the three co-ordinates of each point in the curved surface represent, for any given mass of carbonic acid, a pressure, a temperature, and a volume, which can co-exist in that mass.

"In Dr Andrews' diagram of curves, published in his paper in the *Transactions of the Royal Society* for 1869, p. 583, the experimental results, for each of several temperatures experimented on, are combined in the form of a plane curved line referred to two axes of rectangular co-ordinates. The curved surface in the model is obtained by placing those curved lines with their planes parallel to one another, and separated by intervals proportional to the differences of the temperatures to which the curves severally belong, and with the origins of co-ordinates of the curves situated in a straight line perpendicular to their planes, and with the axes of co-ordinates of all of them parallel in pairs to one

VII. *Clerk Maxwell's Thermodynamic Model.* Lent by Professor Crum Brown

The model shown was one constructed by Maxwell and given by him to the late Professor Chrystal, whose family presented it to the present owner.

Instead of using Professor James Thomson's more obvious co-ordinates, pressure, volume, and temperature (or p, v, t), Professor Willard Gibbs suggested the use of the quantities volume, energy, and entropy, as the rectangular co-ordinates of a surface, and pointed out how the thermodynamic properties of a substance in its solid, liquid, or gaseous states, or in conditions in which these states co-existed, could be indicated by the geometrical properties of such a surface. Maxwell was the first to construct this thermodynamic surface for an arbitary substance and to show clearly how isothermal and isopiestic lines could be drawn upon it.

In the model the volume is measured to the east of the vertical plane of no volume ; energy is measured to the north ; and entropy is measured down.

The red lines are isothermals.

The blue lines are isopiestics.

The simple shadow method by which these can be drawn when the surface is given is explained in Maxwell's *Theory of Heat* (chap. xii.).

The pressure and temperature of the state represented by a point on the surface are represented by the direction of the normal to the surface at the point. Hence, if a plane touches the surface in two or more points, these points represent states of the substance in which the temperature and pressure are the same.

" There is one position of the tangent plane in which it touches the surface in three points. These points represent the solid, liquid, and gaseous states of the substance when the temperature and pressure are such that the three states can exist together. . . .

" From this position of the tangent plane it may roll on the primitive surface in three directions so as in each case to touch it at two points."

another, and by cutting the curved surface out so as to pass through those curved lines smoothly or evenly.*

" The curved surface so obtained exhibits in a very obvious way the remarkable phenomena of the voluminal conditions at and near the critical point of temperature and pressure in comparison with the voluminal conditions throughout other parts of the indefinite range of gradually varying temperatures and pressures. This curved surface also helps to afford a clear view of the nature and meaning of the continuity of the liquid and gaseous states of matter. It does so by its own obvious continuity throughout the expanse to which it might be extended round the outside of the critical point in receding from the range of the points of pressure and temperature where an abrupt change of volume can occur by gasification or condensation. On the curved surface in the model, Dr Andrews' curves for the temperatures 13°·1, 21°·5, 31°·1, 35°·5, and 48°·1 centigrade, from which it was constructed, are shown drawn in their proper places. The model admits of easily exhibiting in due relation to one another a second set of curves in which each curve would be for a constant pressure, and in which the co-ordinates would represent temperatures and corresponding volumes. It serves generally as an aid towards bringing the whole subject clearly before the mind."

* " For the practical execution of this, it is well to commence with a rectangular block of wood, and then carefully to pare it down, applying, from time to time, the various curves as templets to it, and proceeding according to the general methods followed in a shipbuilder's modelling room in cutting out small models of ships according to curves laid down on paper as cross-sections of the required model at various places in its length."

The lines on the surface traced out by these pairs of corresponding points are marked green on the diagram. They give the conditions under which the substance begins to pass from any one of the three states (gaseous, liquid, solid) into either of the other two.

The critical point is where the pairs of corresponding points on the tangent plane coalesce into one as the plane rolls round its line of double touch.

VIII. Closed Linkages. By Colonel R. L. Hippisley, C.B., R.E.

The linkages which form the subject of this exhibit consist of a number of identical three-bar mechanisms in different phases of their motion about two fixed pivots O and O′. If OABO′ denotes the deformable quadrilateral, the several three-bar mechanisms are connected together, so that the point A of one is joined to the point B of its neighbour by a bar of length equal to AB. The whole forms a deformable framework, having one degree of freedom. They are really generalisations of an idea derived from an article by Arnold Emch in the *Annals of Mathematics*, series 2, vol. i. (1900), and are fully described in the *Proc. Lond. Math. Soc.*, series 2, vol. xi. part i.

If a sufficient number of these three-bar linkages are connected together, the last A point may just fall short of the first, or it may overlap it, or it may coincide with it. In the first two cases the gap or overlap is a variable quantity, depending upon the phase of deformation, having two maxima and two minima in one complete revolution of the framework. If, however, the two points coincide in any position, they will coincide in all positions. The linkage is then said to "close." Now this closure can generally be effected by slightly altering the distance between the pivots O and O′, but the calculation of the distance when the number of linkages exceeds three is very laborious. Owing, however, to the well-known fact that all the variables in a three-bar linkage can be rationally expressed in terms of an elliptic parameter, and since the elliptic functions, by reason of their double periodicity, are admirably adapted to problems of closure, the calculation of the *conditions* of closure becomes quite simple; that is to say, the relation, though indeterminate in form and admitting of an infinity of solutions, gives the lengths of all four bars of the linkage if the lengths of two of them are assumed. The method by which this is effected is fully described in the article in the *Proc. Lond. Math. Soc.* above quoted.

If we denote the lengths of the links OA, AB, BO′, O′O by a, b, c, and d, the length d can be adjusted so that

$$(1)\quad a+b>c+d$$
$$(2)\quad a+b<c+d$$
$$(3)\quad a+b=c+d$$

and still be within the conditions of closure. In the first case the link c will continue to revolve after having assumed a position in prolongation of OO′.

In the second case it will not reach this position at all ; and in the third, when it does reach it, all the links will lie together in one straight line.

If in addition to condition (1) we have also $b+c>a+d$, then the a link will continue to revolve after lying in the prolongation of O′O, so that the whole framework, assumed to close, will revolve continuously through a complete circle till it comes back to its original position.

In the second case the link c, after reaching a certain position, which is fixed by a and b lying in a straight line, will turn back, while a continues to revolve. a will go on until it reaches a position in which b and c lie together, and will then also turn back. The corresponding a and c bars of the connected linkages, on reaching these positions, will behave in a similar manner ; and the whole mechanism moves in the peculiar manner which may roughly be described as that of a paper cone, pressed flat, whose two sheets are made to slide over one another continuously round the two creases ; and it is remarkable that one should be able to imitate such a motion by a linkage, especially as the flattened angle of the cone may be, and generally is, greater than 180°.

The third case when $a+b=c+d$ is interesting, because, as can readily be seen, a, b, c, and d cannot lie together in one straight line until all the other a, b, c, d links belonging to the connected linkages are there also. Instead, therefore, of the framework remaining closed when it has once closed, this one opens and shuts like a fan, or, in other words, cannot be closed in the ordinary sense. Moreover, when it is shut up, there is an ambiguity in its motion, for it can either open the other way, which an ordinary fan cannot, or it can return backwards to its original open state. The reason of this apparent contradiction to the *dictum* " once closed, always closed " lies in the fact that the modulus of the elliptic functions becomes unity, and the functions themselves degenerate into hyperbolic functions, which have no real period, and therefore permanent closure cannot be obtained. Moreover, the locus of any carried point, rigidly connected to the traversing link b, has an extra double point over and above the three ordinary nodes and the two triple points at infinity ; and the curve, which is ordinarily a bicursal tricircular sextic, becomes unicursal. The variables are, therefore, no longer expressible in terms of an elliptic parameter. The extra double point occurs at the *dead* point, when the bars are all together in one straight line, where the motion has two degrees of freedom instead of one.

There are in all eight different classes of closed linkages, which arise from the various ways in which the relative lengths of the links can be arranged. If we write

$$\begin{array}{lll} \alpha_1=(b+c)^2, & \alpha_2=(c+a)^2, & \alpha_3=(a+b)^2, \\ \beta_1=(a+d)^2, & \beta_2=(b+d)^2, & \beta_3=(c+d)^2, \\ \gamma_1=(b-c)^2, & \gamma_2=(c-d)^2, & \gamma_3=(a-b)^2, \\ \delta_1=(a-d)^2, & \delta_2=(b-d)^2 & \delta_3=(c-d)^2, \end{array}$$

then these classes are as follows :—

Class.	Relative Magnitudes.			Links which Rotate.	Number of Branches.
1	$\alpha_1\beta_1\gamma_1\delta_1$	$\alpha_2\beta_2\gamma_2\delta_2$	$\beta_3\alpha_3\delta_3\gamma_3$	...	1
2	$\alpha_1\beta_1\gamma_1\delta_1$	$\beta_2\alpha_2\delta_2\gamma_2$	$\alpha_3\beta_3\gamma_3\delta_3$	...	1
3	$\alpha_1\beta_1\delta_1\gamma_1$	$\alpha_2\beta_2\delta_2\gamma_2$	$\alpha_3\beta_3\delta_3\gamma_3$	a b c	2
4	$\alpha_1\beta_1\delta_1\gamma_1$	$\beta_2\alpha_2\gamma_2\delta_2$	$\beta_3\alpha_3\gamma_3\delta_3$	a	2
5	$\beta_1\alpha_1\gamma_1\delta_1$	$\beta_2\alpha_2\gamma_2\delta_2$	$\alpha_3\beta_3\delta_3\gamma_3$	c	2
6	$\beta_1\alpha_1\gamma_1\delta_1$	$\alpha_2\beta_2\delta_2\gamma_2$	$\beta_3\alpha_3\gamma_3\delta_3$	b	2
7	$\beta_1\alpha_1\delta_1\gamma_1$	$\alpha_2\beta_2\gamma_2\delta_2$	$\alpha_3\beta_3\gamma_3\delta_3$	...	1
8	$\beta_1\alpha_1\delta_1\gamma_1$	$\beta_2\alpha_2\delta_2\gamma_2$	$\beta_3\alpha_3\delta_3\gamma_3$	...	1

The ones in which any link rotates are called "bipartite," because the locus of a carried point is then a bipartite curve. The ones in which no link rotates, but simply oscillates between limits, are called "unipartite" for the opposite reason. In the bipartite linkages it is impossible for the links to assume the "crossed" position without disjointing the mechanism. In the unipartite ones such a transition is possible and essential.

Class 1 gives the flattened cone effect described above. There are in reality two flattened cones, with coincident axes, one inside the other, whose apices point in the same direction, and whose circular edges are connected at certain intervals by bars. The analogy of the flattened cone is not perfect, because there has to be a certain elasticity in the material of the two sheets which will admit of its parts moving with a variable angular velocity about its apex.

Class 7 is simply the reflection of Class 1 in a line perpendicular to OO′.

Class 2 has two flattened cones, but the apices point inwards.

Class 8 has two with the apices pointing outwards.

All the above are unipartite.

Class 3 is the simplest. There is no cone effect; it is two wheels without felloes, whose spokes move with variable relative velocities, each spoke of one wheel being connected to a spoke of the other by a bar.

Class 4 is a beam engine with crank and connecting rod.

Class 5 is its reflection in a line perpendicular to OO′.

Class 6 is again two flattened cones, but with non-coincident axes.

The last four classes are bipartite; all of them have another part which is the reflection in OO′.

The closing of these linkages depends upon a principle which was originally due to Cayley, and appears in the *Phil. Trans. Roy. Soc.*, 1861, p. 225. If we denote the variable angles AOO′ and ABO′ by ξ and ω, and the square of the diagonal O′A by x, then

$$x=a^2-2ad\cos\xi+d^2=b^2-2bc\cos\omega+c^2,$$

where it will be noted that x must lie between the greatest of $(a-d)^2$ and

$(b-c)^2$, and the least of $(a+d)^2$ and $(b+c)^2$, as is seen by giving ξ and ω their maximum and minimum values. From these we get

$$\cos\xi=\frac{a^2+d^2-x}{2ad}, \qquad \cos\omega=\frac{b^2+c^2-x}{2bc};$$

which is the same thing as

$$\cos\xi=\frac{\frac{1}{2}(\beta_1+\delta_1)-x}{\frac{1}{2}(\beta_1-\delta_1)}, \qquad \cos\omega=\frac{\frac{1}{2}(\alpha_1+\gamma_1)-x}{\frac{1}{2}(\alpha_1-\gamma_1)};$$

or, as we prefer to write it,

$$\cos\xi=\frac{\beta_1-x-(x-\delta_1)}{\beta_1-\delta_1}, \qquad \cos\omega=\frac{\alpha_1-x-(x-\gamma_1)}{\alpha_1-\gamma_1};$$

and therefore

$$\sin\xi=\frac{2\sqrt{(\beta_1-x)(x-\delta_1)}}{\beta_1-\delta_1}, \qquad \sin\omega=\frac{2\sqrt{(\alpha_1-x)(x-\gamma_1)}}{\alpha_1-\gamma_1}.$$

The form of these immediately suggests that if we take the elliptic integral

$$u=\int_\gamma dx/\sqrt{\mathrm{X}},$$

where X is the quartic function $(\alpha_1-x)\,(\beta_1-x)\,(x-\gamma_1)\,(x-\delta_1)$, and where $\alpha_1>\beta_1>x>\gamma_1>\delta_1$ (which is the case when we are considering Class I, as will be seen from the table), we can express α_1-x, β_1-x, $x-\gamma_1$, $x-\delta_1$, and therefore the cosines and sines of ξ and ω, rationally in terms of Mu, where M stands for $\frac{1}{2}\sqrt{(\alpha_1-\gamma_1)(\beta_1-\delta_1)}$. Putting M$u=\theta$, we can therefore express x rationally as an elliptic function of θ. We shall find in fact that

$$x=\frac{\gamma_1(\beta_1-\delta_1)-\delta_1(\beta_1-\gamma_1)dn^2\theta}{\beta_1-\delta_1-(\beta_1-\gamma_1)dn^2\theta},$$

and

$$k^2=\frac{(\beta_1-\gamma_1)(\alpha_1-\delta_1)}{(\alpha_1-\gamma_1)(\beta_1-\delta_1)}.$$

Now Cayley's principle asserts that if θ_1 and θ_2 are the arguments referring to O′A$_1$ and O′A$_2$ respectively, between θ_1 and θ_2 there exists the relation

$$\theta_2-\theta_1=\mu,$$

where μ is a constant. This can easily be seen to be the case, because, if we differentiate the equation for $\cos\xi$, we get

$$d\xi_1=\frac{2dx_1}{(\beta_1-\delta_1)\sin\xi_1}=\frac{dx_1}{\sqrt{(\beta_1-x_1)(x_1-\delta_1)}},$$

and therefore

$$\frac{d\xi_1}{\sin\omega_1}=\tfrac{1}{2}(\alpha_1-\gamma_1)\frac{dx_1}{\sqrt{\mathrm{X}_1}},$$

and similarly

$$\frac{d\xi_2}{\sin\omega_2}=\tfrac{1}{2}(\alpha_1-\gamma_1)\frac{dx_2}{\sqrt{\mathrm{X}_2}};$$

where the subscripts 1 and 2 refer to the two values of the variables for the two positions of the points A_1 and A_2.

Now it is shown by Emch in his article in the *Annals of Mathematics* that the small displacements of A are proportional to $\sin \omega$, for the instantaneous centre of rotation of the arm A_1B_1 is the point where OA_1 and $O'B_1$ intersect; and if P be this point, the displacement of A_1 is to the displacement of B_1 as PA_1 is to PB_1, that is, as $\sin \omega_1$ is to $\sin OA_1B_1$. Similarly, if Q be the point where OA_2 and $O'B_1$ intersect, the displacement of A_2 is to the displacement of B_1 as QA_2 is to QB_1, or as $\sin \omega_2$ is to $\sin OA_2B_1$; but OA_2B_1 and OA_1B_1 are equal, hence we have

$$\frac{d\xi_1}{\sin \omega_1} = \frac{d\xi_2}{\sin \omega_2},$$

or

$$\frac{dx_2}{\sqrt{X_2}} - \frac{dx_1}{\sqrt{X_1}} = 0;$$

and, integrating,

$$\theta_2 - \theta_1 = \mu, \text{ a constant.}$$

There is of course the same constant interval between θ_3 and θ_2, and between θ_4 and θ_3, and so on. If, therefore, n linkages are joined together in the manner described, and the $(n+1)^{\text{th}}$ position of A coincides with the first, so that x_{n+1} is the same as x_1, θ_{n+1} must differ from θ_1 by some multiple of 4K. But $\theta_{n+1} - \theta_1 = n\mu$, so that in order to close μ must be made equal to $\dfrac{4r\text{K}}{n}$.

Now the elliptic functions of μ can be expressed in terms of the lengths a, b, c, and d. For we can place the linkage in such a position that the value of $\cos \xi$ becomes known. For instance, in Class 1, when a and b are in prolongation of one another, and c at its furthest position, then

$$\cos \xi = \frac{(a+b)^2 + d^2 - c^2}{2d(a+b)},$$

and, since $x = a^2 - 2ad \cos \xi + d^2$, and also $\dfrac{\gamma_1(\beta_1-\delta_1) - \delta_1(\beta_1-\gamma_1)sn^2\theta}{\beta_1-\delta_1-(\beta_1-\gamma_1)sn^2\theta}$,

we readily find, after a little reduction, that

$$sn^2\theta = \frac{\beta_2-\delta_2}{\alpha_2-\delta_2}, \quad cn^2\theta = \frac{\alpha_2-\beta_2}{\alpha_2-\delta_2}, \quad dn^2\theta = \frac{\alpha_2-\beta_2}{\alpha_2-\gamma_2}.$$

But this θ is $\frac{1}{2}\mu$, for, as A_1 approaches this position from one side, A_n is approaching it from the other, and then $\theta_n = 4\text{K} - \theta_1$. But as $\theta_n - \theta_1 = (n-1)\mu = 4\text{K} - \mu$, $2\theta = \mu$. If therefore we calculate from Legendre's tables, or otherwise, the values of $1/sn^2\dfrac{2\text{K}}{n}$ and $1 - dn^2\dfrac{2\text{K}}{n}$, and call them p and q, we can put

$$\frac{\alpha_2-\delta_2}{\beta_2-\delta_2} = p \quad \text{and} \quad \frac{\beta_2-\gamma_2}{\alpha_2-\gamma_2} = q;$$

or

$$a^2 - b^2 + c^2 - d^2 + 2ac + 2bd = 4pbd,$$
$$-a^2 + b^2 - c^2 + d^2 + 2ac + 2bd = 4qac;$$

therefore

$$ac = \frac{p-1}{1-q} bd,$$

and

$$a^2 + c^2 = b^2 + d^2 + \frac{2\{p(1-q) - q(p-1)\}}{1-q} bd;$$

and now, assigning arbitrary values to b and d, we can obtain the corresponding values of a and c, which will ensure the closing of the linkage.

One of the linkages in the exhibit is illustrative of another principle, which was made the subject of a dissertation by Dr Otto Bolduan in 1908 (*Zur Theorie der Uebergeschlossenen Gelenkmechanismen*). If the diagram of the triple generation of the three-bar curve given in Cayley's article in the *Proceedings of the London Mathematical Society*, vol. vii. (1875, 1876), be adapted to the case when the three foci form an equilateral triangle, the three different linkages which generate the same curve can then be shifted bodily until they all work on the same base. If they are then designated by OA_1B_1O', OB_2C_2O', and OC_3A_3O', the addition of three links, B_1B_2, C_2C_3, A_3A_1, of lengths respectively equal to OA_1, OB_2, and OC_3, will ensure the movements of the three linkages being in the same relative phases as they were when joined up in Cayley's diagram. The apices P_1, P_2, and P_3 of equilateral triangles described on A_1B_1, B_2C_2, and C_3A_3 will then describe the same three-bar curve, but displaced about the centre of the focal circle at angles of 120° and 240°.

Now, it can easily be shown that the vertices P_1, P_2, P_3 always form the vertices of another equilateral triangle of variable size. They can be fixed in any one of their positions by pivots; and if now the pivots at O and O′ are freed from constraint, the whole mechanism will be found to have one degree of freedom. It will be seen to consist of three 3-bar linkages, $P_1B_1B_2P_2$, $P_2C_2C_3P_3$, and $P_3A_3A_1P_1$, whose rotating links are rigidly connected in pairs at a fixed angle, *i.e.* P_1A_1 with P_1B_1, P_2B_2 with P_2C_2, and P_3C_3 with P_3A_3. Dr Bolduan has treated this subject from a point of view of greater generality, and has determined the conditions under which three such rigidly connected linkages can exist and at the same time have one degree of freedom.

EXHIBIT

Group of eight linkages in illustration of the previous article. Lent by Colonel R. L. HIPPISLEY, C.B., R.E.

IX. A Double-Four Mechanism. By G. T. BENNETT, F.R.S.

MECHANISMS of the double-four type are quadruply singular in possessing one degree of freedom. They were first discussed by Kempe ("Conjugate Four-piece Linkages," *Proc. Lond. Math. Soc.*, 1878, vol. ix. pp. 133–147), who gave five different species, and afterwards by Darboux, "Recherches sur un système articulé, *Bulletin des Sciences Math.*, 2 série, t. iii., 1879, pp. 151–192.

For a detailed account of the species illustrated by the model see *Proceedings of the Cambridge Philosophical Society*, vol. xvii. pp. 391–401, 1914.

X. Models of the Four-piece Skew Linkage, having Hinge-lines neither Parallel nor Concurrent. By G. T. Bennett, F.R.S.

COMPOUND derivatives; particularly (i) twelve-piece mechanism with two degrees of freedom, (ii) skew double fours, (iii) articulated hyperboloid of revolution, (iv) deformable pseudospherical surface.[1]

Four Groups of Models. Lent by Professor J. E. A. Steggall, M.A.

XI. Four Deformable Models of Surfaces of the Second Degree

THESE are made with universal joints at the intersections of the generating lines, and show very beautifully that the deformation of a ruled quadric into a confocal leaves the lengths of all segments of generating lines unaltered. The proof of this property was proposed by Sir George Greenhill in the Mathematical Tripos of 1878. Cayley and others have written on it.

XII. System of Six-Coloured Cubes

This set of cubes was invented by Major P. A. MacMahon. They are coloured in every possible six-coloured way, and are thirty in number. Certain interesting questions in arrangement arise in connection with the set.

XIII. Projections of the Six Regular Four-Dimensioned Solids

This is a representation, with the dissection of certain parts, of the six regular four-dimensioned solids as shown by their projections in common space. The general idea is perhaps best grasped by considering that while one projection, on a plane, of a cube consists of two similar and similarly situated squares with their corresponding corners joined, the projection in space of the eight-celled rectangular solid consists of two similar cubes similarly treated.

XIV. Miscellaneous Group

(1) *Planigraph*—This is based on the property referred to in XI.

It is clear that two rods, joined by three others with flexible joints, admit of such freedom that they always form a portion of some member of a confocal system: and it is easy to see that if any fourth point be taken on each rod so as to make with the three named an equi-harmonic range on that rod, the joining line of these points is of constant length. If, then, A, B, C, D . . . be considered fixed on one rod, and P, Q, R, S . . . correspond to them on the other, P, Q, R, S . . . will describe spheres with centres

[1] Vide *Engineering*, p. 777, 1903, and *Proc. Lond. Math. Soc.*, vol. xiii. pp. 151–173, 1913.

A, B, C, D . . . If now D be taken at infinity, *i.e.* $\{ABC\infty\}=\{PQRS\}$, S will describe a plane. This is shown by the motion of the terminal point S of the movable rod PQRS.

(2) is an interesting model to show the passage of a twisted curve through a straight line.

(3) gives a system of hyperbolic paraboloids constructed by four equal rods rigidly fixed at two opposite corners, but free at the other two joints. The deformation admissible here should be compared with that in the case of XI.

(4) is a link-work formed by drawing three parallels to the sides from a point within the triangle: it possesses a kind of poristic property, such that the angular points of the original deformed triangle form a triangle of constant shape. The proof of this is a pleasing exercise in the geometry of vectors.

(5) is a small model to illustrate a peculiar method of tracing a quadric from two of its focal conics with the assistance of a stretched string. This is analogous to the description of a confocal curve by means of an endless stretched string passing round any curve of the confocal system.

(6) is a model of a surface which indicates the nature of the roots of the equation $x^5+10ax^3+5bx+c=0$, a, b, c being the co-ordinates of a point on the surface. For the convenience of a well-conditioned model we actually take the equation to be

$$x^5+2ax^3+5bx+\frac{40c}{3}=0\,;$$

the part of the surface being included between the planes

$$\begin{aligned} a&=\pm 125\cdot 00\\ b&=\pm 125\cdot 00\\ c&=\pm 156\cdot 25. \end{aligned}$$

XV. The Semi-Regular Polyhedra and their Reciprocals.

By D. M. Y. Sommerville, D.Sc.

The semi-regular polyhedra are those whose faces are regular polygons with the same length of edge, and which have the same combination of faces meeting at every vertex. They are inscriptible in a sphere. Apart from two infinite series, the prism and the prismatoid, which are bounded above and below by two regular polygons with n sides, and laterally by squares or equilateral triangles respectively, there are just thirteen semi-regular polyhedra. They are all obtainable from the five regular polyhedra by cutting off corners and edges.

The reciprocal of a polyhedron which is inscribed in a sphere is formed by drawing the tangent planes to the sphere at the vertices. The numbers of its faces, edges, and vertices are the same as the numbers of vertices, edges, and faces respectively of the original polyhedron. The reciprocals of the semi-regular polyhedra have their faces and dihedral angles all alike. Several of them are of interest in crystallography, as representing possible forms of natural crystals.

XVI. Projection-Model of the 600-Cell in Space of Four Dimensions. By D. M. Y. Sommerville, D.Sc.

The 600-cell in space of four dimensions may be called the analogue of the icosahedron in three-dimensional space. It is bounded by 600 congruent regular tetrahedra. The model shows a projection of the figure in space of three dimensions. The centre of projection is one of the vertices, so that one vertex of the projection is at infinity. The edges which proceed to infinity have been omitted from the model. This model is constructed to show the successive zones of vertices which surround any vertex. The edges joining the vertices of each concentric zone are formed of brass wire, while the edges joining two different zones are formed of silk threads, or, in one case, of brass wire painted black. Starting from the centre, we have first an icosahedron, then a dodecahedron. The next is an icosahedron whose vertices are not joined to one another, but the edges connecting it with the preceding zone are of black wire. Next we have a semi-regular polyhedron, called the icosadodecahedron, whose faces are triangles and pentagons ; this forms the mesial zone, and the succeeding zones are the same as those already described.

Six Groups of Models. Exhibit from The Mathematical Laboratory, University of Edinburgh

XVII. Wooden Models

1. Regular four-sided pyramid showing normal section.
2. Oblique three-sided prism, cut obliquely so that it may be transformed into a right prism.
3. Right triangular prism, divisible into three tetrahedra.
4. Oblique hexagonal prism, with right section so that it may be transformed into a right prism.
5. Right circular cylinder with oblique section.
6. Cube which may be transformed into a parallelepiped.
7. Six-sided right pyramid.
8. Six-sided oblique pyramid.
9. Five-sided right pyramid.
10. Right circular cone showing the conic sections.
11. Right circular cone showing two sections through vertex.
12. Sphere with two parallel sections.
13. Regular icosahedron.
14. Regular dodecahedron.
15. Prolate spheroid.
16. Anchor ring with sections.
17. Oblate spheroid.

XVIII. Projective Models

1. Projective model, showing projection of quadrilateral into square.
2. Projective model, showing projection of circle into itself.
3. Projective model, showing projection of circle into ellipse.

XIX. Plaster Models

1. Elliptic paraboloid (sections parallel to principal section shown by lines).
2. Elliptic paraboloid (lines of curvature shown).
3. Hyperbolic paraboloid (lines of curvature shown).
4. Hyperbolic paraboloid (generators shown).
5. Hyperbolic paraboloid (hyperbolic sections shown).
6. Hyperboloid of one sheet (generators shown).
7. Hyperboloid of one sheet (lines of curvature shown).
8. Hyperboloid of two sheets (lines of curvature (including generators) shown).
9. Ellipsoid (lines of curvature shown).
10. Ellipsoid (lines of curvature shown).
11. Elliptic cone (lines of curvature (including generators) shown).
12. Envelope of the geodesic lines on a spheroid.
13. Surface $2xyz-x^2-y^2-z^2+1=0$. (See Allardice, *Proc. E.M.S.*, 1891–2.)
14. Helicoid.
15. Surface $z=xy\dfrac{x^2-y^2}{x^2+y^2}$ for which $\dfrac{\partial^2 z}{\partial x\,.\,\partial y}\neq\dfrac{\partial^2 z}{\partial y\,.\,\partial x}$.
16. Surface of revolution with constant negative curvature (Type—cone)
17. Surface of revolution with constant negative curvature (Type—hyperboloid).
18. Surface of revolution with constant positive curvature.
19. Surface of fourth order with one double line.
20. Plücker's surfaces.

XX. Paper Models

1. Hyperbolic paraboloid.
2. Ellipsoids (two).
3. Hyperboloid of one sheet.
4. Hyperboloid of two sheets.
5. Paraboloid.

   (The above models show circular sections.)
6. Model showing an elliptic point.
7. Model showing a parabolic point.
8. Model showing a hyperbolic point.
9. Set of nine models illustrating singular points on surfaces.

XXI. Thread Models

1. Hyperbolic paraboloid (deformable).
2. Hyperboloid of one sheet (deformable).

3. Developable surface formed by the tangents to a cubic ellipse.
4. Cone of third order.
5. Helicoid.
6. A generalisation of the helicoid.

XXII. Miscellaneous Models

1. Cylindroid (by Professor Peddie).
2. Metal cone (double) to show conic sections.
3. Metal catenoid.

XXIII. Models. Exhibited by Charles Tweedie, M.A.

1. Five thread models of ruled quadrics and cylindroids.
2. Fresnel, Wave Surface (constructed by Schilling, Leipzig).

Three Groups of Models. Exhibited by E. M. Horsburgh, M.A.

XXIV. PARALLEL MOTIONS

Among the most important linkages are those which generate a straight line. These are usually called " Parallel Motions," and are classified as (1) true or (2) approximate.

(1) True Parallel Motions

The best-known example is furnished by the Peaucellier cell. If O, P, Q be three collinear points such that OP.OQ=constant, then P and Q describe inverse curves about the fixed pole O. If P describe a circle through O, then Q will describe its inverse, a straight line.

In this linkage, fig. 1, P and Q are the extremities of a diagonal of a rhombus, the extremities of the other diagonal being joined by equal links to O. The constraining bar and the frame form the seventh and eighth links. The vertices are all pin-jointed.

Hart's Crossed Parallelogram

If ABCD be a pin-jointed crossed parallelogram, fig. 2, such that AC is parallel to BD, and if three collinear points O, P, Q be taken in three bars such as AB, BC, DA respectively, and such that OPQ is always parallel to AC or BD, then OP.OQ=constant. If O be a fixed pole, and if P be constrained by a link to describe a circle through O, Q will describe its inverse, a straight line. The constraining bar and the frame form the fifth and sixth links, so that this linkage has two bars less than the former one.

The Double Kite Mechanism

Let QGDC be a pin-jointed kite, and let C be the centre for the long, and G for the short, arms. Let DCBA be a similar kite, which has double the

linear dimensions of the former one, and in which A is the centre for the long, and C for the short, arms. Further, let the points A, G, D be collinear. This forms a double kite, fig. 3. If OAGQ be a parallelogram, and OQ the frame of the mechanism, then the point B will describe a straight line.

(2) Approximate Parallel Motions

The Scott-Russell Parallel Motion

In the ordinary ellipsograph a rod of constant length slides with its extremities on two rectangular axes. Any point in the rod describes an ellipse, and in particular the middle point C describes a circle, whose centre is the origin O, fig. 4. If, then, this point C be constrained by a link to describe this circle, and if one extremity slide in a straight guide OA, the free end B will generate a straight line whose accuracy depends on the straightness of the guide.

Grasshopper Parallel Motion

There are many modifications of the Scott-Russell parallel motion. In practice it is desirable to replace sliding by turning whenever possible, owing to difficulties due to dead centres and friction. Hence, an approximate straight-line motion is obtained by using a link to constrain one extremity A of the bar to describe an arc of a large circle (an approximate straight line), and another link to constrain any one point D on the bar to describe an arc of a circle, representing as closely as possible the osculating circle of the elliptic arc described by that point, fig. 5. Thus, an approximate straight line is generated by the other extremity B of the bar.

Tchebicheff's Parallel Motion

Let ADB and AEC be the sides, and BC the base of an isosceles triangle, and let DE be parallel to BC. Then a jointed mechanism, fig. 6, may be formed by the links BE, ED, DC, CB. If BC be fixed, and if P be the middle point of DE, then P generates an approximate straight line.

Roberts' Parallel Motion

Let ABP, PCD be two equal equilateral triangles on the same side of the straight line APD, AP and PD being their bases. This figure represents the linkage in its mid-position. Let there be pin-joints at A, B, C, and D, and let BCP be a rigid equilateral triangle, fig. 7. Then P describes an approximate straight line.

Watts' Parallel Motion

This simple four-bar mechanism, fig. 8, is the most important of those approximate parallel motions which are formed by four turning pairs. It consists in its simplest form of two cranks, with the crank-pins joined by a coupler. In the mid-position, if the cranks be horizontal, with a phase difference of 180°, the coupler is vertical. The tracing point P divides the coupler inversely as the lengths of the nearest cranks. The approximate

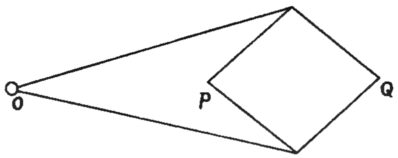


Fig 1

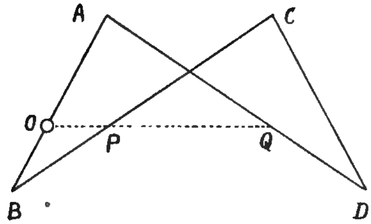


Fig 2

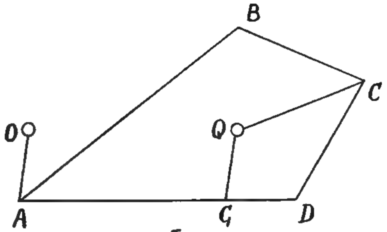


Fig 3

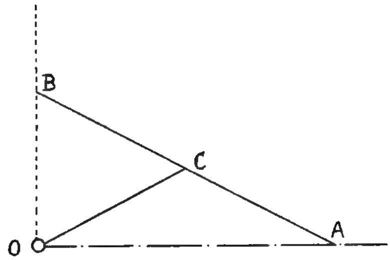


Fig 4

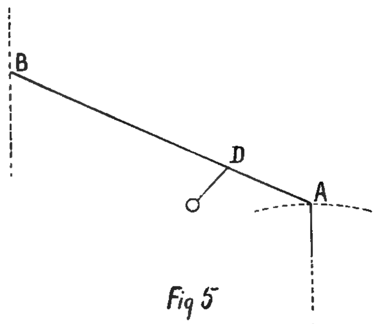


Fig 5

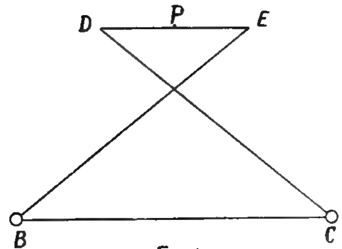


Fig 6

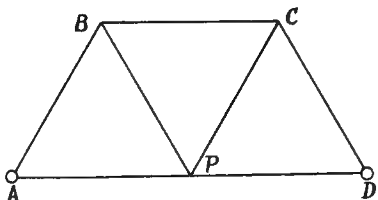


Fig 7

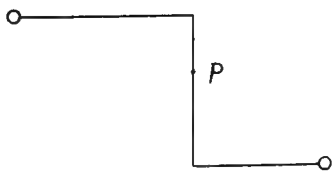


Fig 8

straight line is near the node of the closed path. See "How to Draw a Straight Line," by Kempe.

The importance of parallel motions in the early days of mechanical engineering was due to the difficulty of cutting straight guides for the cross-head, valve spindle, and pump rods of the engines. The necessity for them has now disappeared, owing to the perfection of modern machine tools. They are still of some use, however, as in indicators.

XXV. PANTOGRAPHS

If P, Z, and F be three collinear points, and if P be a fixed pole and the ratio PZ/PF constant, then Z and F will describe similar figures. A linkage which makes use of this property is called a pantograph, and is used for copying diagrams on a larger or smaller scale.

If F be the tracing point, and Z the pencil point, and if Z be nearer to the pole P than F, the copy is a reduction, while if the positions of Z and F be interchanged it becomes an enlargement.

The Eidograph is an improvement on the pantograph, and aims at greater accuracy.

The Skew Pantograph was invented by Sylvester. It enlarges or reduces a given figure and rotates it through a given angle. (See *Nature*, xii. (1875), pp. 168, 214.)

Of recent years the pantograph has been superseded by photography, but the instrument is coming into favour again in the form of the "Precision Pantographs" (see Section G). In these the bars of the linkage are partly suspended by fine wires from the top of a heavy upright standard, while verniers and micrometers are provided for the accurate setting of the links.

XXVI. MISCELLANEOUS

1. The Limaçonograph (Chrystal, *Proc. Roy. Soc. Ed.*, xxiv. 19, 1901).
2. Multisector and Lazy-tongs, for multisecting angular and linear space.
3. Four-bar, and slider crank mechanisms.
4. Simple epicyclic trains.
5. In illustration of simple shear.
6. In illustration of the prismoidal formula.
7. T-linkage for describing equal areas (*Proc. Edin. Math. Soc.*, vol. xxxi.).

SUBSECTION

I. Geometrical "Plastographs" or "Anaglyphs" designed and executed by Mr F. G. Smith, of H.M. Patent Office. By EDWARD M. LANGLEY, M.A.

IN these a stereoscopic effect is produced by viewing bi-coloured diagrams through absorption screens, after the method discovered by W. Rollman and described by him in *Poggendorff's Annalen* for 1853. The method, though used and possibly re-invented by D'Almeida, appears to have attracted little attention, and to have received few applications until used in connection with photography by Duhauron during the years 1891–1895. Naturally, after the publication of sets of such views, the idea of applying the method to the representation of geometrical figures occurred independently to various investigators interested in the representation of solids, among others to M. H. Richard, of Chartres (some of whose designs have been published by Vuibert), and to Mr F. G. Smith, of H.M. Patent Office, whose designs are now shown.

Mr Smith's collection includes: the successive reflection of a ray of light by three mirrors at right angles to one another; sections of a helicoid; interpenetration of prisms; octahedron and cube; Kelvin's 14-face and cube; projection of a quadrilateral into a parallelogram.

II. Models after Max Brückner. By EDWARD M. LANGLEY, M.A.

THESE are reproductions of some of the simpler models figured in *Vielecke und Vielfläche*, and the later work *Über die gleicheckig-gleichflächigen discontinuerlichen und nichtconvexen Polyeder.*

SECTION K

PORTRAITS AND MEDALS

I. Collection of Portraits of Mathematicians, past and present,

in nine quarto volumes. Lent by W. W. ROUSE BALL, M.A.

THE portraits are divided into three groups :—

1st. A general collection—contained in the volumes numbered from 1 to 7.

2nd. A collection of portraits of the more eminent mathematicians and physicists—contained in Volume A.

3rd. A collection of portraits of Professors and University Lecturers in Mathematics at Cambridge—contained in Volume C.

In each of these groups the portraits are arranged in alphabetical order, and in the Catalogue which follows the names are given, accompanied by biographical notes of all the mathematicians whose portraits are included in the several volumes. In the case of many well-known names in Volumes A and C notes have not been necessary.

CATALOGUE

VOLUME I

Count F. Algorotti, 1712–1764. Poet, Mathematician, and Physicist.
D. Algöwer, 1678–1737. Professor of Mathematics at Ulm. Meteorologist.
T. Allen (of London), 1542– .
G. J. Allman, 1824–1904. Professor of Mathematics at Galway, Ireland.
A. M. Ampère, 1775–1836.
A. Anderssen (of Breslau), 1818–1879.
F. Andreossy, 1633–1688.
S. de Angelis, 1623–1697.
P. Anich (of Innsbruck), 1723-1766.
Petrus Apianus (Bienewitz), 1495–1552. Professor of Mathematics and Astronomy at Ingolstadt.
Philippus Apianus, 1531–1589. Professor of Mathematics at Tübingen.
D. F. J. Arago, 1786–1853.
R. Arkwright, 1732–1792. Mechanician.
J. Averranius, 1663–1738. Professor of Laws and Astronomy at Pisa.
A. L. Bacler-Dalbe, 1761–1824. Employed by Napoleon for geodetical surveys.
Lord (Francis) Bacon, 1561–1626.
W. Bagwell (of London), 1593–1659.
J. W. Baier, 1675–1729. Professor of Physics and Mathematics at Altdorf, etc.
J. S. Bailly (of Paris), 1736–1793. (Portrait and specimen of handwriting.)
R. Baltzer, 1818–1887. Professor of Mathematics at Giessen.
John Bansi (of London), 1576– . Astrologer and Chemist.
J. P. Baratier, 1721–1740. Infant Prodigy in Mathematics.
E. W. Barnes (of Cambridge).
F. Barrème, –1703.
L. A. Bartenstein, 1711–1796. Professor of Mathematics at Gotha.
Cosmo Bartholi, 1515– .
E. Bartholinus, 1625–1698. Professor of Geometry and Medicine at Copenhagen.
T. Bartholinus, 1616–1680. Professor of Mathematics at Copenhagen.
J. von Beauchamp, 1752–1801.
J. Beck, 1741–1805. Professor of Mathematics at Vienna.
A. H. Becquerel (of Paris), 1852–1908.
E. Beltrami, 1835–1900. Professor at Rome.
W. W. Beman, 1850– . Professor of Mathematics at Michigan, U.S.A.
J. I. Berghaus (of Münster), 1753–1831.
J. Bernard, 1658–1718.
M. Bernegger, 1582–1640. Professor at Strassburg.
John Bernoulli (II.), 1710–1790. Professor of Mathematics at Bâle.

VOLUME I—*continued*

John Bernoulli (III.), 1744-1807. Astronomer Royal at Berlin.
P. Bertius, 1565-1629. Professor of Philosophy at Leyden, and subsequently of Mathematics at Paris.
W. H. Besant (of Cambridge), 1828- .
R. O. Besthorn (of Copenhagen), 1847- .
E. Betti, 1823-1892. Professor at Pavia.
M. Beuther, 1522-1587. Professor of Mathematics at Greifswald and of History at Strassburg.
H. Beyer (of Frankfort), 1516-1577.
D. Bierens de Haan, 1822-1895. Professor of Mathematics at Leyden.
G. B. Bilfinger, 1693-1750. Physicist. Professor at St Petersburg and Tübingen.
N. Bion (of Paris), 1655-1733. Mechanician and Astronomer.
Jean Baptiste Biot (of Paris), 1774-1862. Astronomer and Physicist.
G. Birkbeck, 1746-1841. Professor of Natural Philosophy at Glasgow.
C. A. Bjerknes, 1825- Professor of Mathematics at Christiania.
Joseph Black, 1728-1799. Professor of Chemistry at Edinburgh.
V. Bobynin, 1849- . Professor of the History of Mathematics at Moscow.
J. E. Bode, 1747-1826. Director of the Observatory, Berlin.
A. Böhm, 1720-1790. Professor of Mathematics at Giessen.
George Boole, 1815-1864. Professor at Queen's College, Cork.
C. W. Borchardt, 1817-1880.
L. A. Bougainville, 1729-1811.
The Hon. Robert Boyle, 1627-1691.
James Bradley, 1692-1762. Savilian Professor of Astronomy at Oxford. Astronomer Royal (1742-1762).
A. von Braunmühl, 1853- . Professor of Mathematics at Munich.
F. Brioschi, 1824-1897. Professor at Pavia and at Milan.
William, Viscount Brouncker, 1620-1684.
N. Bruyant, 1572-1638.
J. du Bucquoy, 1693-1760.
J. C. Burckhardt, 1773-1825. Director of the Observatory, Paris.
W. Burnside (of Cambridge).
J. G. Büsch, 1728-1800. Professor of Mathematics at Hamburg.

VOLUME II

F. Cajori, 1859- . Professor of Mathematics at Tulane and of Physics at Colorado, U.S.A.
J. F. v. B. Calkoen, 1772-1811. Dutch Astronomer.
S. Calvisius, 1555-1615.
N. L. Sadi Carnot, 1796-1832.
J. Carpov, 1699-1768. Professor of Mathematics and Philosophy at Weimar.
J. Casey (of Dublin).
J. Dom. Cassini, 1625-1712. Director of the Observatory, Paris. (Two Portraits.)
L. van Ceulen, 1539-1610. Professor of Mathematics at Leyden.
M. Chemnitz, 1522-1586.
J. P. L. de Chesaux (of Lausanne), 1718-1751. Astronomer.
S. A. Christensen, 1861- . Professor of Mathematics at Odense, Denmark.
George Chrystal, 1851-1911. Professor of Mathematics at Edinburgh.
E. D. Clarke, 1769-1822. Professor of Minerology at Cambridge.
Samuel Clarke (of Cambridge), 1675-1729.
Chr. Clavius, 1537-1612. Professor of Mathematics at Rome.
H. W. Clemm, 1725-1775. Professor of Theology and Mathematics at Tübingen.
Edward Cocker, 1631-1677.
P. Coecke, 1502-1550. Architect, Painter, and Mathematician.
J. de Collas, 1678-1752.
C. M. de la Condamine (of Paris), 1701-1774.
J. A. de Condorcet, 1743-1794.
Luigi Cremona, 1830-1900. Professor at Rome.
Sir William Crookes, 1832- .
J. P. de Crousaz, 1683-1753. Professor of Mathematics and Philosophy at Groningen, etc.
C. Cruciger, 1504-1548. Professor at Wittenberg.
Nicholas Culpepper (of Cambridge), 1616-1653. (Two Portraits.)
S. Curtius (of Nuremberg), 1576-1650.
E. L. W. M. Curtze, 1837- . Professor of Mathematics at Thorn.
J. Dalby, 1744-1824. Professor of Mathematics, R. M. C. Farnham.
John Dalton (of Manchester), 1766-1844.
I. B. N. D. D'Après, 1707-1780.
J. G. Darjes, 1714-1791. Professor of Mathematics at Frankfort.
J. M. L. Dase (of Berlin), 1824-1861.
Leonardo da Vinci, 1452-1519.
John Dee (of Cambridge), 1527-1608.
E. de Jouquières, 1820-1901.
J. De la Lande, 1732-1807. Professor at Paris.
J. B. Delambre, 1749-1822.
P. J. Derivaz, 1711-1772.
S. Dickstein, 1851- . Professor of Mathematics at Warsaw.
J. Ditzel, 1654-1710. Professor of Mathematics at Leipzig.
A. C. Dixon (of Cambridge), 1865- .
J. de N. Dobrzensky, 1631-1697. Professor of Mathematics and Rector of Prague.
John Dollond, 1706-1761.
J. G. Doppelmair (of Nuremberg), 1671-1750.
H. W. Dove, 1803-1879. Professor of Physics at Königsberg and at Berlin.
J. Dryander, 1500-1560. Professor at Marburg.
W. H. Dufour, 1785-1875.
J. Duns Scotus, 1245-1308.
F. P. Ch. Dupin, 1784- . Professor at Paris.
A. Durer, 1471-1528.
F. W. Dyson, Astronomer Royal.
J. J. Ebert, 1737-1805. Professor at Wittenberg.
L. Eickstad, 1596-1660. Professor of Mathematics and Medicine at Danzig.
G. C. Eimmart, 1638-1705. Mathematician and Astronomer.
E. Eisinga (of Friesland), 1744-1823. Astronomer.
R. L. Ellis (of Cambridge), 1817-1859.
J. F. Encke, 1791-1865.
G. Eneström, 1852- .
William Esson, 1838- Savilian Professor of Geometry at Oxford.
C. F. Eversdyk, 1586-1666. Arithmetician.
J. A. Eÿtelwein (of Berlin), 1764-1848. Physicist and Engineer.

VOLUME III

J. Faber (of Paris), 1455–1530 (?). Arithmetician.
Samuel Faber, 1657–1706.
G. A. Fabricius (of Mülhausen and Göttingen), 1589–1645. Physicist.
J. B. Fabricius (of Nuremberg), 1564–1626.
M. Faraday, 1791-1867.
J. Faulhaber (of Ulm), 1580–1635.
A. Favaro, 1847– . Professor of Graphical Statics at Padua.
A. Feist.
James Ferguson, 1710-1776.
John Fernel, 1497–1558. Mathematician and Physician.
N. M. Ferrers (of Cambridge), 1829-1903.
P. Fixlmillner, 1721-1791. Astronomer.
M. Flaccus (of Berlin), 1524–1592. Astrologer and Astronomer.
R. Fludd, 1574-1637. Physicist and Astrologer.
D. Fontana, 1543–1607. Mathematician and Architect.
B. de Fontenelle (of Paris), 1657–1737. Poet, Astronomer, and Philosopher.
Simon Forman (of Cambridge), 1552–1611.
J. B. L. Foucault (of Paris), 1819–1868.
J. Fracastor (of Verona), 1483–1553. Astronomer.
B. Franklin (of U.S.A.), 1706-1790. Physicist.
J. v. Fraunhofer (of Munich), 1787–1826. (Two Portraits.)
A. J. Fresnel, 1788–1827. Physicist.
N. Frischlin, 1547–1590. Professor at Tübingen and Braunschweig.
A. P. Frisi, 1728–1784. Professor of Mathematics at Milan.
G. L. Frobenius (of Hamburg), 1566-1644.
P. Frost (of Cambridge), 1817–1898.
P. Gassendi, 1592-1655. Professor at Paris.
L. Gaucicus (of Padua), 1476–1558. Astronomer and Astrologer.
J. L. Gay-Lussac (of Paris), 1778–1850. Physicist. (Two Portraits.)
P. Geiger (of Zurich), 1569– . Arithmetician.
E. Gelcich, 1854– . Professor of Mathematics at Cattaro, and Director of Naval Instruction in Austria.
J. de Gelder, 1765–1848. Professor of Mathematics at Leyden.
R. Gemma, 1508–1555. Dutch Mathematician, Astronomer, and Physician.
W. Geus (of Nuremberg), 1519– . Astronomer.
J. Willard Gibbs, 1839–1903. Professor of Mathematical Physics at Yale, U.S.A.
J. W. L. Glaisher (of Cambridge), 1848– .
R. Goclenius, 1572-1621. Professor of Mathematics and Physics at Marburg.
W. J. s'Gravesande, 1688–1742. Professor at Leyden.
David Gregory, 1661–1710. Savilian Professor at Oxford.
D. F. Gregory (of Cambridge), 1813-1844.
Olinthus Gregory, 1774–1841.
Gregory (Saint Vincent), 1584–1667. Professor of Mathematics at Prague.
Sir Thomas Gresham (of Cambridge), 1519–1579.
J. F. Griendl (of Nuremberg), –1688. Mathematician and Optician.
Otto von Guericke (of Magdeberg), 1602–1686.
D. Guilelminus (of Padua), 1655–1710. Astronomer.
S. Günther, 1848– . Professor of Mathematics at Ansbach, and of Geography at Munich.
J. Hadley (of London), 1670–1744. Brought the Sextant into general use.
P. M. Hahn (of Würtemberg), 1739–1790. Meteorologist and Astronomer.
Edmund Halley, 1656-1742. Astronomer Royal.
G. B. Halsted, 1853– . Professor of Mathematics at Colerado, U.S.A.
G. A. Hamberger, 1662–1716. Professor of Mathematics and Physics at Jena.
Sir William R. Hamilton (see Vol. A). Photograph of Brougham Bridge, renamed by Hamilton "Quaternion Bridge." Over the Royal Canal three miles from Dublin, two miles from Dunsink; on which Sir W. R. Hamilton cut the i.j.k. of quaternions at the moment of discovery on 16th October 1843.
M. C. Hanov, 1695–1773. Professor at Danzig.
P. A. Hansen, 1795–1874. Director of the Observatory at Gotha.
J. Harrison (of London), 1693–1776.
G. Hartman (of Nuremberg), 1489–1564.
J. Hartwich, 1592– .
E. Hatton, 1664–1716. Arithmetician.
J. L. Hauenreuter, 1548–1618. Professor of Medicine and Mathematics at Strassburg.
C. A. Hausen, 1693–1743. Professor of Mathematics at Wittenberg and at Leipzig.
J. L. Heiberg (of Copenhagen), 1854– .
V. Heins (of Hamburg), 1637–1704.
G. Heinsius, 1709–1769. Professor at Leipzig and at St Petersburg.
M. Hell, 1720–1792. Director of the Observatory at Vienna.
J. Heller, 1518–1590. Professor of Mathematics and Astronomy at Nuremberg.
J. C. L. Hellwig, 1743–1831. Professor of Mathematics at Braunschweig.
G. Henisch (of Augsburg), 1549-1618. Mathematician and Physician.
C. W. Hennert (of Berlin), 1739–1800. Mathematician and Geographer.
J. S. Henslow, 1796–1861. Professor of Mineralogy and, subsequently, of Botany at Cambridge.
J. Herbst, 1642– .
D. Herlicius (of Lübeck), 1557–1636. Astronomer.
F. B. W. Hermann, 1795–1868. Professor of Mathematics and Technology at Munich.
Sir John Herschel, 1792-1871.
William Herschel, 1748–1822. Astronomer.
H. Hertz, 1857–1894. Professor of Physics at Bonn.
J. Hevilius (of Danzig), 1611–1687. Astronomer.
C. Heyden, 1526–1576. Professor at Nuremberg.
Thomas Hill (of Cambridge), –1558.
C. F. Hipp, 1763–1838. Professor of Mathematics at Hamburg.
J. L. Hocker (of Heilsbronn), 1670–1746. Theologian and Mathematician.
James Hodder, fl. 1661.
J. Hoene Wronski, 1778–1853. Author of Works on the Philosophy of Mathematics.
V. Hofmann (of Nuremberg), 1610–1682.
E. B. Holst, 1849– . Professor of Mathematics at Christiania.
J. C. Horner, 1774–1834. Professor of Mathematics at Zürich.
Samuel Horsley (of Cambridge), 1733–1806.
J. J. Huber, 1733–1798. Astronomer at Greenwich, Berlin, and Bâle.
R. W. H. T. Hudson (of Cambridge), 1877–1904.
Sir William Huggins, 1824–1910. (Two Portraits.)
F. Hultsch (of Dresden), 1833– .
Alex. von Humboldt, 1769–1859.
K. Hunrath, 1847– .
Charles Hutton, 1737–1823. Professor at Woolwich.
A. G. Hyperius, 1511–1564. Mathematician and Astronomer. Professor of Theology at Marburg.
M. Imkof, 1758–1817. Professor of Mathematics, Physics, and Chemistry at Munich.

VOLUME IV

J. de Indagine, *circ.* 1560.
F. de P. Jacquier, 1711–1788. Professor of Physics and Mathematics at Rome.
J. W. A. Jäger (of Nuremberg) 1718 - .
C. Jezeler, 1734–1791. Professor of Mathematics, etc., at Schaffhausen.
J. P. Joule (of Manchester), 1818-1889.
J. Junge, 1587-1657. Professor of Mathematics at Giessen, subsequently Rector of Hamburg.
U. Junius, 1670–1726. Professor of Mathematics at Leipzig.
A. G. Kästner, 1719-1800. Professor at Göttingen.
I. Kant, 1724–1804. Professor of Philosophy at Königsberg.
W. J. G. Karsten, 1732–1787. Professor of Physics and Mathemetics at Hälle.
E. L. von Kautenacker.
Lord Kelvin, 1824-1907. (Four Portraits. See also Vol. A.)
C. Kirch (of Berlin), 1694-1740. Astronomer.
G. Kirch (of Berlin), 1639–1710. Astronomer to the King of Prussia.
A. Kircher (of Würzburg), 1602-1680.
G. R. Kirchhoff, 1824–1887. Professor of Physics, etc., at Heidelberg.
H. Klausing, 1675–1745. Professor of Mathematics and Theology at Leipzig.
G. S. Klügel, 1739–1812. Professor of Mathematics and Physics at Helmstadt and Hälle.
C. G. Knott, 1856-
J. M. Köberlein, 1768–1837. Professor of Mathematics at Regensburg.
P. Kolb, 1675–1726. Astronomer at Cape of Good Hope and Neustadt.
J. M. Korabinsky, 1740–1811. Mathematician and Geographer.
G. F. von Kordenbusch, 1731–1802. Professor of Physics and Mathematics at Nuremberg.
S. Kowalevski (of Stockholm), 1853-1891.
G. W. Krafft, 1701–1754. Professor of Mathematics and Physics at St Petersburg and at Tübingen.
J. G. F. Krafft, 1751–1795. Professor of Mathematics at Bayreuth.
N. Kratzer, *circ.* 1528. Clockmaker and Astrologer to Henry VIII. of England.
C. Kreil, 1798-1862. Professor of Astronomy and Physics at Prague and Vienna
J. Kromayer (of Leipzig), 1610-1670. Mathematician and Theologian.
J. E. Kruse (of Hamburg), 1709–1775.
H. Künssberg, 1854– Professor of Mathematics at Dinkelsbühl.
H. Lamb, 1849- . Professor of Mathematics at Adelaide and Manchester.
G. Lamé, 1795–1870. Professor at Paris.
C. Langhausen, 1660–1727. Mathematician. Pastor of Königsberg.
Dionysius Lardner, 1793–1859.
E. Lemoine, 1840–1912.
J. A. Leuneschulus (of Heidelberg), 1619– .
J. Leupold (of Leipzig), 1674-1727.
W. J. Lewis, 1847– . Professor of Mineralogy at Cambridge.
G. F. A. de L'Hospital, 1661 1704.
G. C. Lichtenberg, 1742–1799. Professor at Göttingen. Physicist and Astronomer.
F. H. Lichtscheid, 1662–1707.
J. G. Liebknecht, 1679–1749. Professor of Theology and Mathematics at Giessen.
B. A. von Lindenau (of Altenberg), 1780–1854.
C. L. von Littrow (of Vienna), 1811–1877.
J. J. von Littrow, 1781–1840. Professor of Astronomy and Director of the Observatory at Vienna.
G. D. Liveing, 1861– Professor of Chemistry at Cambridge.
Sir Oliver J. Lodge, 1851– .
J. C. Löhe, 1723–1768. Professor of Mathematics and Physics at Nuremberg. Theologian.
A. Lonicerus, 1528–1586. Professor of Mathematics at Nuremberg.
J. C. Ludeman (of Hamburg), 1685–1757. Astrologer.
J. Lütkemann, 1608–1655. Professor of Mathematics and Physics at Greifswald. Theologian.
R. Lulle, 1235–1315. Astrologer and Alchemist.
J. Lulofs, 1711-1768. Professor of Mathematics and Astronomy at Leyden.

VOLUME V

E. Mack, 1838– . Professor at Prague and Vienna.
J. H. Mädler, 1794–1874. Professor of Astronomy and Director of the Observatory at Dorpat.
M. Maestlin, 1550–1631. Professor of Mathematics at Tübingen. Galileo and Kepler were his pupils.
J. A. Maginus, 1555–1617. Professor of Mathematics at Bologna.
C. J. Malmsten (of Upsala), 1814–1886.
V. Mandey, 1646–1702.
P. Mansion, 1844– . Professor of Mathematics at Ghent.
H. M. Marcard, 1747–1817.
A. Marcel, 1672–1748.
A. Marchetti, 1633–1714. Professor of Mathematics at Pisa.
J. F. Mari (of Paris), 1738-1801.
M. Martini (of Berlin).
Baron F. Masères (of Cambridge), 1731-1824.
N. Maskelyne, 1732–1811. Astronomer Royal.
C. Mason, 1698-1770. Professor of Geology at Cambridge. Physicist.
P. L. M. de Maupertuis (of Berlin), 1698–1759.
F. Maurolycus, 1494–1575. Professor at Messina.
M. F. Maury, 1806–1873. Director of the Observatory of Washington, U.S.A. Subsequently Professor of Physics at Lexington.
F. T. Mayer, 1723–1762. Professor at Göttingen.
D. Melanderhjelm, 1726–1810. Professor of Astronomy at Upsala.
M. Mersenne (of Paris), 1588–1648.
C. Meurer, 1558-1616. Professor of Mathematics and Physics at Leipzig.
J. A. C. Michelsen, 1747–1797. Professor of Mathematics at Berlin.
W. H. Miller, 1801–1880. Professor of Mineralogy at Cambridge, 1832–1880.
H. Minkowski, 1864-1909.
B. Mithobius, 1504–1565. Professor of Mathematics and Medicine at Marburg.
A. F. Möbius, 1790–1868. Professor of Astronomy at Leipzig.
F. N. M. Moigno, 1804–1884. Professor at Paris.
G. Moll, 1785–1838. Professor of Mathematics and Physics at Utrecht.
J. B. van Mons, 1765–1842. Professor of Physics and Chemistry at Brussels.
O. Montalbani, 1601–1671. Professor of Mathematics, Medicine, etc., at Bologna.
G. Montanari, 1633–1687. Professor of Mathematics at Bologna, and of Astronomy at Padua.
J. A. von Monteiro (of Lisbon), 1758– . Physicist and Chemist.
J. E. Montucla, 1725–1799.
John Hamilton Moore, *circ.* 1775.
Sir Samuel Morland (of Cambridge), 1625–1696.
J. H. Müller, 1671-1731. Director of Observatory of Nuremberg, and Professor of Mathematics and Physics at Altdorf.
J. H. J. Müller, 1809–1875. Professor of Physics at Freiburg.
N. Mulerius, 1564–1630. Professor of Medicine and Mathematics at Groningen. (Two Portraits.)

VOLUME V—*continued*

J. de Munck (of Middleburg), 1687–1760. Astronomer to William IV. of Holland.
J. de Muralt, 1645–1733. Professor of Mathematics and Physics at Zürich. State Physician.
R. Murphy (of Cambridge), 1806–1843.
P. van Musschenbroek, 1692–1761. Professor of Mathematics and Physics successively at Duisburg, Utrecht, and Leyden.
E. Narducci (of Rome), 1832–1893.
P. Naudé, junior, 1684–1745. Professor of Mathematics at Berlin.
E. Netto, 1846– Professor of Mathematics at Giessen.
J. Neudorffer, senior (of Nuremberg), 1497–1563.
J. F. Niceron, 1613–1646. Author of various works on Optics, and one on Ciphers.
G. Nicolai, 1726–1793. Professor of Mathematics at Padua.
D. R. v. Nierop (of Hoorn, Holland), seventeenth century. Astronomer and Mathematician.
B. Nieuwentyt (of Purmerende), 1654–1718.
P. Nieuwland, 1764–1794. Professor of Mathematics, etc., at Leyden.
N. Nye, 1624– .
J. C. Odontius, 1580–1626. Professor of Mathematics at Altdorf.
H. W. M. Olbers, 1758–1840. Astronomer.
B. Oriani (of Milan), 1753–1832.
D. Origanus, 1558–1628. Professor of Mathematics and Philosophy at Frankfort.
William Paley (of Cambridge), 1743–1805.
J. G. Palitzsch, 1732–1786. Astronomer.
P. S. Pallas, 1741–1811. Physicist, Geographer, Traveller.
G. H. Paricius (of Ratisbon), 1675–1725.
Stephen Parkinson (of Cambridge), 1823–1889.
E. Pascal, 1865– . Professor at Naples.
M. Pasor, 1599–1658. Professor of Mathematics at Heidelberg and Groningen.
N. C. F. de Peiresc, 1580–1637. Physicist, Philosopher, and Man of Letters.
J. F. Penther, 1693–1749. Professor of Mathematics at Göttingen.
S. J. Perry (of Stoneyhurst), 1833–1889. Astronomer.
C. Pescheck (of Zittau), 1676–1747.
N. Petri, *circ.* 1596.
J. F. Pfeffinger, 1667–1730. Professor of Mathematics at Lüneburg.
A. Piccolomini, 1508–1578. Mathematician, Astronomer, and Philosopher.
M. A. Pictet, 1752–1825. Professor of Physics at Geneva. Physician.
Julius Plücker, 1801–1868. Professor of Mathematics and Physics at Bonn.
J. F. Polack, 1700–1771. Professor of Law and Mathematics at Frankfort.
G. Poleni, 1683–1761. Professor of Philosophy, Astronomy, and Mathematics at Padua.
G. della Porta, 1558–1615. Optician and Physicist.
J. C. Posner, 1673–1718. Professor of Physics and Rhetoric at Jena.
William Postel, 1510–1581. Professor of Mathematics at Paris.
J. H. Poynting, 1852–1914. Professor of Physics at Birmingham.
L. Praalder, 1706–1796. Lector at Utrecht.
J. Prætorius, 1537–1616. Professor of Mathematics at Altdorf.
Joseph Priestley, 1733–1804.
R. A. Proctor (of Cambridge), 1837–1888.

VOLUME VI

P. Ramus, 1515–1572. Professor at Paris.
W. J. Macquorn Rankine, 1820–1872. Professor of Engineering at Glasgow.
R. A. F. de Réaumur, 1683–1757.
L. W. von Regler, –1792. Mathematician, Surveyor, Soldier.
P. Riccardi, 1828– . Professor of Geometry at Bologna.
J. Riccati, 1676–1754.
A. Riese, 1489–1559. Arithmetician (Portrait and specimen of handwriting).
F. Rivard, 1697–1778. Philosopher, Mathematician.
J. Rohault (of Paris), 1620–1675. Mathematician and Physicist.
J. B. von Rohr, 1688–1742. Mathematician and Chemist.
G. Rollenhagen (of Magdeburg), 1542–1609. Astronomer and Astrologer.
W. C. Röntgen, 1845– . Professor of Physics at Würzburg and Munich.
A. Rossignol (of Paris), 1590–1673.
Count Rumford, 1753–1815.
Sir Ernest Rutherford, 1871– . (Two Portraits.)
E. Sang, 1805–1890.
P. Sarpi, 1552–1623. Mathematician, Scholar, and Theologian.
Sir Henry Savile (of Oxford), 1549–1622.
P. Saxe, 1591–1625. Professor of Mathematics at Altdorf.
J. J. Scaliger, 1540–1609. "The Father of Chronology." Professor at Leyden.
Sir Charles Scarborough (of Cambridge), 1616–1693.
E. C. J. Schering, 1833– .
J. J. Scheuchzer, 1672–1733. Professor of Mathematics and Physics at Zürich.
G. V. Schiaparelli, 1835–1910.
W. Schickard, 1592–1635. Professor of Mathematics and Hebrew at Tübingen.
S. Schinz, 1734–1784. Professor of Mathematics and Physics at Zürich.
E. Schmid, 1570–1637. Professor of Mathematics and Greek at Wittemberg.
J. A. Schmid, 1652–1726. Professor of Mathematics and Theology at Helmstadt.
J. Schoner (of Nuremberg), 1477–1547.
M. Schoockius, 1614–1655. Physicist and Scholar. Professor at Utrecht, Deventer, Groningen, and Frankfort.
Frans van Schooten, –1660. Professor of Mathematics at Leyden.
C. Schorer (of Memmingen), 1618–1674.
E. O. Schreckenfuchs, 1511–1579. Professor of Mathematics and Hebrew at Tübingen.
J. F. L. Schröder (of Utrecht), 1774–1845.
J. H. Shröter, 1745–1816. Astronomer.
Sir A. Schuster (of Manchester), 1851– .
J. C. Schwab (of Stuttgart), 1743–1821. Astronomer, Mathematician, and Philosopher.
D. Schwenter, 1585–1636. Professor of Mathematics and Hebrew at Altdorf.
A. Secchi (of Rome), 1818–1878. Astronomer.
T. J. J. See, 1866– .
J. A. von Segner, 1704–1777. Professor of Physics and Mathematics at Göttingen.
C. Segre, 1863– . Professor of Higher Geometry at Turin.
John Sems, 1573–1600. (Two Portraits.)
C. E. Senff, 1810–1849. Professor of Mathematics at Dorpat.
J. F. Sentelet, –1829. Professor of Mathematics and Physics at Louvain.
A. Sharpe, 1653–1742.
W. N. Shaw, 1854– .
J. Simler, 1530–1576.

VOLUME VI—*continued*

S. Slominski (of Bialystock), *circ.* 1820.
R. Snell, 1547–1613. Professor of Mathematics and Hebrew at Leyden.
Willebrod Snell, 1591–1626.
Mary F. Somerville, 1780-1872.
A. Spole, 1630-1699. Professor of Mathematics at Upsala.
J. Stadius, 1527–1579. Professor ot Mathematics at Löwen and Paris.
J. S. Stedler, *circ.* 1680. Professor of Mathematics at Erlangen.
M. Steinschneider (of Berlin), 1816– .
A. Stern. Polish Mathematician.
J. Stöffler, 1452–1531. Professor of Mathematics at Tübingen.
G. Johnstone Stoney, 1826–1911.
Æ. Strauch, junior, 1632–1682. Professor of Mathematics and History at Wittemberg. Pastor.
C. A. von Struensce, 1735–1804. Professor of Mathematics and Hebrew at Halle, and of Mathematics at Liegnitz.
F. G. W. Struve (of Pulkowa), 1793–1864.
N. Struyck (of Amsterdam). Astronomer.
J. C. Sturm, 1635-1703. Professor of Mathematics and Physics at Altdorf.
L. C. Sturm, 1669–1719. Professor of Mathematics at Frankfort.
S. G. Succov, 1721–1786. Professor of Mathematics, Philosophy, and Physics at Erlangen.
J. G. Sulzer, 1720-1779. Professor of Mathematics at Berlin.
H. Suter, 1848– Professor of Mathematics at Zürich.
J. H. van Swinden, 1746–1823. Professor of Mathematics, etc., at Amsterdam.

VOLUME VII

J. Taisner, 1509–1563. Astrologer to Charles V. Astronomer.
P. G. Tait, 1831–1901. Professor of Natural Philosophy at Edinburgh.
D. Talius, –1583. Professor of Hebrew and Mathematics at Altdorf.
P. Tannery (of Paris), 1843–1904.
F. G. Teixeira, 1851– . Professor of Analysis at Coimbra and Porto.
J. N. Tetens, 1736–1807. Professor of Physics, Mathematics, and Philosophy at Kiel.
B. G. Teubner, 1784–1856. (Two Portraits.)
P. E. Tigurinus, 1563– .
J. Tischberger (of Nuremberg), 1715–1793.
Felix Tisserand, 1847– . Director of the Paris Observatory.
J. Toaldo, 1719–1797. Professor of Astronomy at Padua.
I. Todhunter (of Cambridge), 1820-1884. Author of numerous text-books.
Cuthbert Tonstall (of Cambridge), 1474-1559.
G. Toulli (of Verona), 1721-1781. Geometer.
E. Torricelli, 1608–1647.
A. Trew, 1597–1669. Professor of Mathematics, Physics, and Astronomy at Altdorf.
J. G. Trigler, 1614–1678.
W. P. Turnbull (of Cambridge).
J. Tyndall, 1820–1893. Professor of Physics at the Royal Institution, London.
G. Vacca, 1872– .
G. Valentin (of Berlin), 1848– .
P. Valentino.
E. Hildericus von Varel, 1533–1599. Professor of Mathematics at Jena, etc.
P. Varignon, 1654–1722. Professor of Mathematics at Paris.
G. Vicuna, 1840–1890. Professor of Mathematical Physics at Madrid.
A. des Vignolles, 1649–1744.
G. Vivanti, 1859– . Professor of the Calculus at Pavia.
E. Vogel, 1829–1856. Astronomer—worked in Africa.
J. H. Voigt, 1613– .
A. Volta (Count), 1745–1827. Professor of Physics at Pavia and at Padua.
R. C. Wagner, seventeenth century.
H. Wahn, *circ.* 1730.
G. T. Walker, 1868- .
W. Walton (of Cambridge), 1813–1901.
Seth Ward, 1617–1689. Savilian Professor of Astronomy at Oxford.
Richard Watson (of Cambridge), 1737–1814. *Moderator* in 1763, when he introduced the system of "Classes."
James Watt, 1736–1819.
G. W. Wedel (of Jena), 1645–1721. Physicist, Chemist, and Physician.
E. Weigel, 1625–1699. Professor of Mathematics at Jena.
J. Weisbach, 1806–1871. Professor of Mathematics at Freiberg.
H. Weissenborn, 1830–1896. Professor of Mathematics at Eisenach.
E. Welper, 1590–1616. Professor of Mathematics at Strassburg.
J. Werner (of Nuremberg), 1468-1528. Astronomer and Mathematician.
Sir C. Wheatstone, 1802–1875. Professor at London.
William Whewell (of Cambridge), 1794-1866. (Two Portraits.)
C. J. von Wiebeking, 1762–1842.
J. B. Wiedeburg, 1687–1766. Professor of Mathematics at Helmstadt and at Jena.
J. Wilkins (of Cambridge), 1614–1672. Author of "Mercury" on Ciphers, etc.
C. J. von Wolf, 1679–1754. Professor of Mathematics and Philosophy at Halle and Marburg.
R. Wolf, 1816–1893. Professor of Astronomy at Berne, and of Astronomy and Mathematics at Zürich.
W. H. Wollaston (of Cambridge), 1766–1828.
R. Woltman, 1757–1837.
James Wood (of Cambridge), 1760 1839.
Sir Christopher Wren, 1632–1723. Professor of Astronomy at Oxford.
Thomas Wright (of Durham), 1711–1786.
F. X. von Wulfen, 1728–1805. Professor of Mathematics, etc., at Klagenfürt.
J. P. v. Wurzelbau, 1651–1725. Astronomer and Mathematician.
W. Xylander, 1532–1570. Professor of Mathematics and Greek at Heidelberg.
Thomas Young (of Cambridge), 1773–1829.
J. Zabarella, 1533-1589. Professor at Padua—wrote on Perpetual Motion.
F. X. von Zack (of Gotha), 1754–1832. Astronomer.
O. Zanotti-Bianco, 1852– . Professor of Geometry at Turin.
A. Zendrini (of Venice), 1763–1849. Professor of Mathematics at Venice.
H. G. Zeuthen, 1839– . Professor of Mathematics at Copenhagen.

VOLUME A

N. H. Abel, 1802-1829.
P. E. Appell, 1858- .
E. Beltrami, 1835-1900.
Daniel Bernoulli, 1700-1782.
James Bernoulli, 1654-1705.
John Bernoulli, 1667-1748.
F. W. Bessel, 1784-1846.
M. B. Cantor, 1829- . (Portrait and Letter.)
G. Cardan, 1501-1576. (Two Portraits.)
L. N. M. Carnot, 1753-1823.
A. L. Cauchy, 1789-1857.
B. Cavalieri, 1598-1647.
Henry Cavendish, 1731-1810.
A. C. Clairaut, 1713-1765.
R. F. A. Clebsch, 1833-1872. (Two Portraits.)
W. K. Clifford, 1845-1879.
Nicholas Copernicus, 1473-1543.
J. D'Alembert, 1717-1783. (Two Portraits.)
J. G. Darboux, 1842-
Abraham de Moivre, 1667-1754.
Augustus de Morgan, 1806-1871. (Two Portraits.)
R. Descartes, 1596-1650. (Two Portraits.)
P. G. J. Lejeune Dirichlet, 1805-1859. (Two Portraits.)
F. G. Eisenstein, 1823-1852.
L. Euler, 1707-1783. (Two Portraits.)
P. de Fermat, 1601-1665. (Two Portraits.)
John Flamsteed, 1646-1719.
J. Fourier, 1768-1830.
J. L. Fuchs, 1833-1902.
Galileo, 1564-1642. (Two Portraits.)
E. Galois, 1811-1832.
C. F. Gauss, 1777-1855. (Three Portraits and Facsimile of Gauss's Diary.)
H. G. Grassmann, 1809-1872. (Two Portraits.)
James Gregory, 1638-1675.
G. H. Halphen, 1844-1889.
Sir William R. Hamilton, 1805-1865.
H. von Helmholtz, 1821-1894.
Ch. Hermite, 1822-1901.
C. Huygens, 1629-1695.
C. G. J. Jacobi, 1804-1851.
Lord Kelvin, 1824-1907. (Two Portraits and a Letter.)
J. Kepler, 1571-1630.
F. C. Klein, 1849- .
L. Kronecker, 1823-1891. (Two Portraits.)
J. L. Lagrange, 1736-1813. (Two Portraits.)
P. S. Laplace, 1749-1827.
A. M. Legendre, 1752-1833.
G. W. Leibnitz, 1646-1717.
Leonardo Fibonacci, 1175-1250? (Authority doubtful.)
U. J. J. Leverrier, 1811-1877.
M. Sophus Lie, 1842-1899. (Two Portraits.)
N. I. Lobatschewsky, 1793-1856.
G. Loria, 1862- .
Colin Maclaurin, 1698-1746.
G. Mercator, 1512-1594.
G. Monge, 1746-1818.
John Napier, 1550-1617. (Two Portraits.)
S. Newcomb, 1835-1909.
M. Nöther, 1844- .
W. Oughtred, 1574-1660.
B. Pascal, 1623-1662.
C. E. Picard, 1856.
H. Poincaré, 1854-1912.
S. D. Poisson, 1781-1840.
Regiomontanus, 1436-1476.
G. F. B. Riemann, 1826-1866. (Two Portraits.)
George Salmon, 1819-1904. (Two Portraits.)
Henry J. S. Smith, 1826-1883.
J. Steiner, 1796-1863.
J. J. Sylvester, 1814-1897.
Tartaglia, 1506-1559.
Brook Taylor, 1685-1731.
P. L. Tchebychef, 1821-1894.
Tycho Brahe, 1546-1601. (Two Portraits with Facsimile of Autograph and Pictures.)
F. Vieta, 1540-1603.
J. Wallis, 1616-1703.
W. Weber, 1804-1891.
K. Weierstrass, 1815-1897.

VOLUME C.—CAMBRIDGE PORTRAITS

This volume contains portraits of the Lucasian, Plumian, Lowndean, Jacksonian, Sadlerian, Cavendish, and Engineering Professors, the University Lecturers in Mathematics, and two or three Private Tutors who were specially prominent. The following is the list:—

J. C. Adams, 1819-1892. (Portrait and Bookmark.)
Sir George B. Airy, 1801-1892. (Portrait and a Letter.)
Charles Babbage, 1792-1871. (Two Portraits.)
H. F. Baker.
Sir Robert S. Ball.
Isaac Barrow, 1630-1677.
A. Berry.
T. J. I'a. Bromwich.
A. Cayley, 1821-1895. (Two Portraits and a Letter.)
James Challis, 1803-1882.
John Colson, 1680-1760.
Roger Cotes, 1682-1716.
Sir G. H. Darwin, 1845-1912. (Two Portraits.)
John Dawson, 1734-1820.
A. S. Eddington.
Sir J. A. Ewing, 1855- .
W. Farish, 1759-1837.
A. R. Forsyth, 1858- (Three Portraits.)
R. T. Glazebrook.
G. H. Hardy.
E. W. Hobson, 1856- . (Three Portraits.)
W. Hopkins, 1805-1866.
B. Hopkinson, 1874-
J. H. Jeans.
Joshua King, 1798-1857.
Sir Joseph Larmor.
W. Lax, 1761-1836.
J. G. Leathem.
Roger Long, 1680-1770.
A. E. H. Love. (Two Portraits.)
W. H. Macaulay.
H. M. Macdonald.
G. B. Mathews.
J. Clerk Maxwell, 1831-1879.
Isaac Milner, 1751-1820.
H. F. Newall, 1857- .
Sir Isaac Newton, 1642-1727.
George Peacock, 1791-1858.
R. Pendlebury, 1847-1902.
Lord Rayleigh, 1842- (Two Portraits and a Letter.)
H. W. Richmond.
E. J. Routh, 1831-1907.
Nicholas Saunderson, 1682-1739.
Anthony Shepherd, 1722-1795.
John Smith, 1711-1795.
Robert Smith, 1689-1768. (Two Portraits.)
Sir George G. Stokes, 1819-1903.
James Stuart, 1843- .
Sir Joseph J. Thomson, 1856- .
T. Turton, 1780-1864.
Samuel Vince, 1754-1821.
Edward Waring, 1736-1798.
W. Whiston, 1667-1752.
E. T. Whittaker.
Robert Willis, 1800-1875.
Robert Woodhouse. (Autograph only.)

II. Newton Medals and Token Coinage. Lent by W. W. Rouse Ball, M.A.

The collection consists of eight medals struck to commemorate Sir Isaac Newton. A full description of the medals accompanies the case in which they are placed. There is also a collection of the Newton token coinage issued in 1793–4. These are set on pivots in an ebony frame with silver mounts.

III. Engravings of John Napier of Merchiston

(*a*) 5 inches × 3 inches. Engraved by L. Stewart from an original painting in the College Library, Edinburgh. John Napier of Murchiston, Inventor of the Logarithms. Edinburgh, published by A. Constable & Co.

(*b*) 5 inches × $3\frac{1}{4}$ inches. John Napier of Murchiston, Inventor of Logarithms. London, William Darton, 58 Holborn Hill.

(*c*) $1\frac{5}{8}$ inches × $1\frac{6}{8}$ inches. Napier.

(*d*) $10\frac{1}{4}$ inches × $8\frac{1}{4}$ inches. John Napier. From an engraving by Stewart, after an original painting in Edinburgh. (Three copies.)

(*e*) $3\frac{1}{4}$ inches × $5\frac{3}{8}$ inches. R. Cooper, sculp[t]. Napier of Murchiston, from a rare print by Delaram. Published by Charles and Henry Baldwin, Newgate Street.

Framed Engravings

(1) John Napier of Merchistoun, $9\frac{1}{4}$ inches × $7\frac{1}{4}$ inches. Lent by Archibald Hewat, F.F.A.

(2) *a*, *d*, *e* lent by E. M. Horsburgh, M.A.

(3) *c* lent by George Smith, M.A.

(4) Napier, Gregory, Maclaurin, and others, from the Mathematical Laboratory, University of Edinburgh.

(5) Napier, various portraits, lent by W. Rae Macdonald, F.F.A.

(6) Portrait of J. Hoene Wronski. Lent by S. Dickstein.

SECTION L

MISCELLANEOUS AND LATE EXHIBITS

I. Two Sets of "Napier's Bones" or Numbering Rods

(1) Lent by C. J. WOODWARD.
(2) Lent by JOHN R. FINDLAY, D.L.

II. Five-Figure Logarithmic Tables. Two Volumes

(1) For Chemists. (2) Ordinary. These are side-indexed. Lent by C. J. WOODWARD.

III. Exhibit by Professor S. Dickstein

HOENE WRONSKI, *Canons de Logarithmes*. Published in Paris in 1827; republished in Polish by Professor S. Dickstein, Warsaw, 1890.

A very ingenious set of tables for finding the logarithms of numbers to four, five, six, or seven places. Each table occupies one sheet of numbers suitably arranged, and there are six sheets in all. The principle is based on the approximate identity

$$\log (a+m+z)=\log a+\Delta_m \log a+z\Delta_1 \log (a+m)$$

where

$$\Delta_m \log a=\log (a+m)-\log a$$

and

$$\Delta_1 \log (a+m)=\log (a+m+z)-\log (a+m).$$

In these expressions a, m, z are suitably chosen parts of the given number, and are in rapidly diminishing magnitude. The marvellous compactness of the tables is, of course, counterbalanced by the necessity of having to build up most of the logarithms by a process which requires both thought and time.

IV. "T.I.M." and "UNITAS" Calculating Machines

THE " T.I.M." Single Slide Calculating Machine claims to be a great advance on the old style of Arithmometer. The chief advantages specified are quietness, simplicity of construction, ease of turning the handle, rapidity, and optional partial clearance.

The " Unitas " Double Slide Calculating Machine claims to be a great advance in calculating machines. With this a series of multiplications may be worked on the middle slide, each being shown separately on this slide, whilst the final result is shown on the top slide. By separation of the levers it is possible to check on one slide what is being done on the other.

V. A New Form of Harmonic Synthetiser.

By J. R. MILNE, D.Sc.

SOME nine years ago, Professor Chrystal, who was then investigating the " seiches " of the Scottish lochs, asked the author if he could design a special form of harmonic synthetiser to assist in the work of analysing the curves obtained. The intention was to use the apparatus to draw a large number of different curves of known harmonic constituents to serve as standards of comparison. It was hoped that in this way the general species of a limnograph curve might be recognised merely by inspection, thus saving much exploratory calculation.

Of course, many synthetisers were then in existence, but not one in which it was possible to alter gradually the period and amplitude of the constituent harmonics *while the machine was in motion* ; and as it was considered likely that such gradual changes occur in the case of seiches, it was necessary that pattern curves should be at hand exhibiting the results.

In regard to the construction of the apparatus, it was decided to make trial of the principle of having *every* moving part turn about a centre. This form of construction has not only the merit of being relatively inexpensive, but it also possesses the great advantage of giving rise to much less friction than is produced by the use of sliding parts.

No difficulty was experienced in building a machine on these lines, and its working has shown the principle to be quite satisfactory. For full details, reference must be made to a paper published in the Royal Society of Edinburgh's *Proceedings* for 1906, page 208, and entitled " A New Form of Harmonic Synthetiser," but the leading points may be briefly explained. The pen of the instrument, constrained to move in a vertical straight line by means of a parallel-motion linkage, was attached to the end of a wire which served to sum up the motion of the various harmonics. For that purpose the wire was alternately led up and down between fixed pulleys F and movable pulleys M, the latter being attached to the " harmonic wheels " H. The distance FM being about 30 inches, and the eccentricity of the pulleys on the harmonic

wheels half an inch, it is shown by the mathematical analysis given in the paper cited that the deviation from true simple harmonic motion due to the finite length of FM is insensible. The pen P actuated by the summation wire moves up and down in front of an upright strip of paper travelling horizontally and constantly unwound from a roll by means of an electric motor, which at the same time drives the harmonic wheels. In order that the periodic time of the latter may be variable at will, it suffices to connect each to the motor through the intermediary of two coned pulleys; the belt connecting each pair of pulleys can then be slid along them, by means of a suitable guide, so as to alter

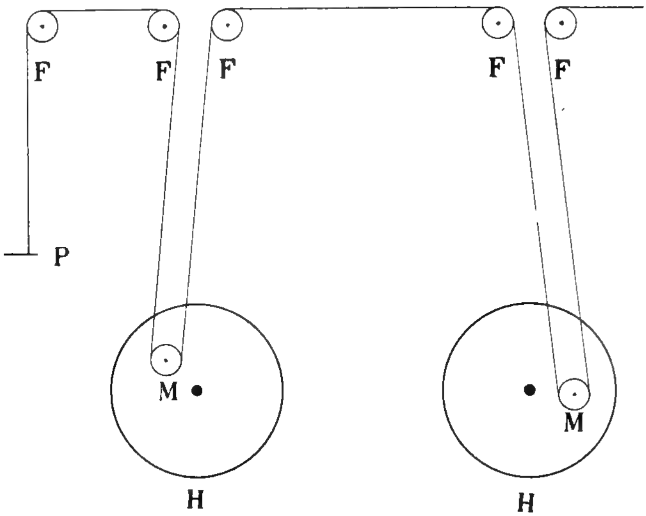


FIG. 1.

the gear ratio of its harmonic wheel to the motor as desired. Such alterations can, of course, be made while the machine is running, thus fulfilling one of the required conditions.

The other condition, namely, that the amplitude of the harmonic should also be variable at will during motion, is fulfilled thus. The harmonic wheel is made in duplicate, the two wheels being connected by a crown wheel so as to form the differential gear now familiar to everyone because of its use on motor cars.[1] The effect of this duplication of the harmonic wheel is to set up *two* simple harmonic motions in the wire of equal period and amplitude, but differing in phase by an amount which depends on the position of the crown wheel. By displacing the axle of the latter through 90°, the phase difference of the harmonics can be altered from 0° to 180°.

In symbols, the effect of the two simple harmonic motions is

$$a \cos \frac{2\pi t}{p} + a \cos \left(\frac{2\pi t}{p} + \alpha\right),$$

where a is the common amplitude, p the common period, and α the phase difference.

[1] See Dunkerley's *Mechanism*, paragraph headed "Jack-in-the-Box Mechanism."

But the above expression is equal to

$$2a \cos\frac{\alpha}{2} \cos\left(\frac{2\pi t}{p}+\frac{\alpha}{2}\right),$$

a simple harmonic motion of the required period, with an amplitude which can be varied at will from 0 to $2a$ by varying α; and this alteration can of course be carried out while the machine is in motion. (A method of avoiding the concurrent alteration of phase—if that be objectionable—is explained in the paper quoted.)

VI. New Table of Natural Sines

ATTENTION may be drawn to a Table of Natural Sines just published by Mrs E. GIFFORD, in which for the first time the "advance" of the argument is one second of arc, and which has a range of from 0° to 90°.

The values of the sines to 10″ were copied from the *Opus Palatinum* of Rheticus, and then the values of the sines to 1″ were interpolated by the Thomas Calculating Machine, each value being copied to ten places. Tables of differences are provided in the margin of each page for the purpose of interpolating to fractions of a second. (See p. 54.)

VII. The R.H.S. Calculator

THIS is a form of cylindrical slide rule. The most recent form of this instrument is lent by the designer, Professor ROBERT H. SMITH. It is accompanied by instructions for use. (See p. 172.)

VIII. Exhibit by E. M. Wedderburn, D.Sc.

LIMNOGRAMS of Seiches. Seiche Normal Curves. Temperature Seiche Records.

IX. A Mechanism for the Solution of an Equation of the *n*th Degree. By Professor W. PEDDIE, D.Sc.

X. Early Form of Slide Rule, with two Slides

Lent by A. G. BURGESS, M.A.

XI. Description of Auguste Beghin's Special Model of Calculating Rule

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XII. Table of Compound Interest at $\frac{1}{8}$ per cent., and of Antilogarithms to sixty figures to base 1·00125. By JOSEPH J. STUCKEY, M.A., A.I.A., Adelaide. Printed by Unwin Brothers, Limited, 27 Pilgrim Street, London, E.C., and Woking, Surrey

THE main part of the Table gives values of 1·00125, or, in other words, the antilogarithm to the base 1·00125. The tables, being worked out to a small base, lend themselves to interpolation, and enable the calculator to obtain values of compound interest functions with little trouble.

XIII. Percentage Theodolite and Percentage Compass and Percentage Trigonometry or Plane Trigonometry reduced to Arithmetic. By JOHN C. FERGUSSON, M.Inst.C.E.

XIV. Portrait of John Napier. Lent by Sir ALEXANDER L. NAPIER

Section M

LIST OF CONTRIBUTORS AND EXHIBITORS

Adams, A. C., A.M.I.M.E., 1 Old Smithhills, Paisley, **F** (4).

Anderson, Brigadier-General F. J., 4 Treboir Road, Kensington, London, S.W., **F** (3).

Andoyer, H., Professeur à la Faculté des Sciences de l'Université de Paris, Membre du Bureau des Longitudes, Paris, **C**, I.

Archimedes and Colt, Calculating Machines, 4 Albert Square, Manchester (Agent, Alex. Angus, 61 Frederick Street, Edinburgh), **D**, I. (1).

Ball, W. W. Rouse, M.A., Fellow of Trinity College, Cambridge, **K**, I. and II.

Barkla, Charles G., D.Sc., F.R.S., Professor of Natural Philosophy, University of Edinburgh, **B**, I.(5).

Baxandall, D., South Kensington Museum, London, W., **F** (1).

Bell, Herbert, M.A., B.Sc., Assistant in Natural Philosophy, University of Edinburgh, **C**, III. (*a*).

Bennett, G. T., M.A., F.R.S., Fellow of Emmanuel College, Cambridge, **I**, IX. and X.

Blackmore, Mrs, Forden House, Moretonhampstead, **C**, IV. (1).

British Calculators, "Brical," Invicta Works, Belfast Road, Stoke Newington, London, N., **D**, I. (3).

Brown, Professor A. Crum, M.D., D.Sc., LL.D., F.R.S., 8 Belgrave Crescent, Edinburgh, **I**, (*a*) and I.–VII.

Brunsviga Calculating Machines, Wellington Chambers, 46 Cannon Street, London, E.C. (G. M. Müller, Sales Manager), **D**, I. (4).

Burgess, A. G., M.A., F.R.S.E., 64 Strathearn Road, Edinburgh, **L**, X.

Burroughs Adding and Listing Machine, E. Hawkins, Sales Manager, Cannon Street, London, E.C., **D**, I. (5).

Carse, George A., M.A., D.Sc., F.R.S.E., Lecturer in Natural Philosophy, Univ. of Edinburgh, **G**, II. (*a*), III. (*a*), VI. (*a*).

Colt's Calculators, 4 Albert Square, Manchester, **D**, I. (2).

Comptometer (see Felt and Tarrant), **D**, I. (6).

Coradi, G., Zürich, **G**, II. (1), III. (1), XI. (1) and (2).

Dantzig, Town Library of, **C**, I.

Davis, John, & Sons, All Saints Works, Derby, **F** (16).

Dickstein, Professor S., Warsaw, **L**, III.

Dunlop, Mrs Mercer, 23 Campbell Avenue, Murrayfield, Edinburgh, **B**, II. (3).

Education, Board of, London, **B**, V.

Erskine-Murray, J. R. See Murray.

Esnouf, Auguste, A.C.G.I., Port Louis, Mauritius, **F** (9).

Euklid Calculating Machine (see Mercedes-Euklid).

Evans, Lewis; Russells, near Watford, **B**, II. (1) ; **F** (2).

Faber, A. W., **F** (17).

Felt & Tarrant Manufacturing Co., Imperial House, Kingsway, London, W.C., **D**, I. (6).

Findlay, John R., M.A., D.L., J.P., 27 Drumsheugh Gardens, Edinburgh, **B**, IV. ; **C**, I. ; **L**, I.

Forrester, Miss Catherine, 30 Snowdon Place, Stirling, **B**, I. (8).

Gibb, David, M.A., B.Sc., F.R.S.E., Lecturer in Mathematics, University of Edinburgh, **G**, VIII., IX., X. (*a*).

Gibson, G. A., M.A., LL.D., F.R.S.E., Professor of Mathematics, University of Glasgow, **A** (*a*).

Gifford, Mrs E., Oaklands, Chard, **L**, VI.

Gregorson, A. M., W.S., Ardtornish, Colinton, **B**, II. (2).

Hammond Typewriter Co., 50 Queen Victoria Street, London, E.C., **D**, III. (1).

Harvey, Major W. F., I.M.S., Director Pasteur Institute of India, Rasauli, **E** (2).

Henderson, Adam, F.S.A. (Scot.), The Library, University of Glasgow, **B**, VIII.
Hewat, Archibald, F.F.A., F.I.A., F.R.S.E., 13 Eton Terrace, Edinburgh, **K**, III. (1).
Hilger, Adam (see Millionaire Calculating Machine).
Hippisley, Colonel R. L., C.B., R.E., 106 Queen's Gate, South Kensington, London, S.W., **K**, VIII.
Horsburgh, Ellice M., M.A., B.Sc., Assoc. M.Inst.C.E., F.R.S.E., Lecturer in Technical Mathematics, University of Edinburgh, **D**, I. (10) ; **F** (10) ; **G**, II. (2) ; **H**, I. ; **I**, XXIV.–XXVI ; **K**, III. (2).
Horsburgh, the Rev. Andrew, M.A., Lynton, St Mary Church, Torquay, **B**, VII.
Hudson, T. C., B.A., H.M. Nautical Almanac Office, 3 Verulam Buildings, Gray's Inn, W.C. See Burroughs Adding Machine, **D**, II. (1).

Jardine, W., M.A., B.Sc., 40 Albion Road, Edinburgh, **D**, I. (8) ; **H**, I. (1).

Kelvin, Bottomley & Baird, 16–20 Cambridge Street, Glasgow, **G**, VII. (2).
Knott, Cargill G., D.Sc., General Secretary R.S.E., Lecturer on Applied Mathematics in the University of Edinburgh, and formerly Professor of Physics in the Imperial University of Tokio, Japan, **E**, (*a*) and (1) ; **C**, II.

Langley, Edward M., M.A., Bedford Modern School, **I**, Subsection I. and II.
Layton, C. & E., 56 Farringdon Street, London, E.C., **D**, I. (7).
Lilly, W. E., M.A., M.A.I., D.Sc., M.I.C.E. Ireland, Engineering School, Trinity College, Dublin, **F** (5).

Macdonald, W. R., F.F.A., 4 Wester Coates Avenue, Edinburgh, **B**, III. ; **K**, III. (6).
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